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LECTURE: (1)

Subject: matrices
Level: First
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Chapter One

Consider an arbitrary system of equation in unknown as:

AX = B ……………………………………………………………(1)

$$\left.\begin{array}{l} a_{r1}X_1 + a_{12}X_2 + a_B X_3 + ... + a_{1n}X_n \\ ail\chi_1 + ai_2\chi_2 + ai_3\chi 3 + ... + ain\chi n = b_1 \\ a21\chi_1 + a22\chi 2 + a23\chi 3 + ... + a2n\chi n = b2 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + ... + a_{2n}X_n \\ am1\chi 1 + am2X2 + 9m3\chi 3 + ... + a2n\chi n = bm \\ a_{m1}X_1 + a_{m2}X_2 + a_{m3}X_3 + ... + a_{mn}X_n \end{array}\right\} ……………………………(2)$$

The coefficient of the variables and constant terms can be put in the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a2 & a22 & a2n \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix}_{mxn} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \\ \chi n \end{pmatrix}_{nx1} \begin{pmatrix} b1 \\ b2 \\ \\ bm \end{pmatrix}_{mx1} ……………………………………..(3)$$

Let the form

$$\begin{pmatrix} a_u a_{12} a_{1n} \\ \\ a_{21} a_{22} a_{2n} \\ \\ am_1 am_2 amn \end{pmatrix} = A = (a_{i1}) ……………………………………………...(4)$$

Is called (mxn) matrix and donated this matrix by:

[aij] i = 1, 2,……m and j = 1, 2,…..n.

We say that is an (mxn) matrix or تكملة

The matrix of order (mxn) it has m rows and n columns.

For example the first row is (a_{11}, a12, a_{1n})

And the first column is $\begin{pmatrix} a_{12} \\ a_{11} \\ a_{21} \\ a_{m1} \end{pmatrix}$

(aij) denote the element of matrix. Lying in the i – th row and j – th column, and we call this element as the (i,j) - th element of this matrix

Also $\begin{pmatrix} \chi 1 \\ \chi_2 \\ \\ \chi_n \end{pmatrix}_{nx1}$ is (nx1) [n rows and columns]

$\begin{pmatrix} b1 \\ b2 \\ \\ bm \end{pmatrix}_{mx}$ Is (mx1) [m rows and 1 column]

Sub – Matrix:

Let A be matrix in (4) then the sub-matrix of A is another matrix of A denoted by deleting rows and (or) column of A.

Let A = $\begin{pmatrix} 1\ 2\ 3 \\ 4\ 5\ 6 \\ 7\ 8\ 9 \end{pmatrix}$

Find the sub-matrix of A with order (2×3) any sub-matrix of A denoted by deleting any row of A $\begin{pmatrix} 1\ 2\ 3 \\ 4\ 5\ 6 \end{pmatrix}, \begin{pmatrix} 1\ 2\ 3 \\ 7\ 8\ 9 \end{pmatrix}, \begin{pmatrix} 4\ 5\ 6 \\ 7\ 8\ 9 \end{pmatrix}$

Definition 1.1**:**

Tow (mxn) matrices A = [aij] (mxn) and B = [bij] (mxn) are said to be equal if and only if:

aij = bij for i = 1,2…..m and j = 1,2….n

Thus two matrices are equal if and only if:

i. They have the same dimension, and
ii. All their corresponding elements are equal for example:

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & 0(7) & -2+1 \\ 3 & \frac{20}{4} & 2 \end{bmatrix}$$

Definition 1.2

If A = [aij] mxn and B = [bij] mxn are mxn matrix their sum is the mxn matrix A+B $=[aij + bij]mxn$.

In other words if two matrices have the same dimension, they may be added by addition corresponding elements. For example if:

$$A = \begin{pmatrix} 2 & -7 \\ -3 & 4 \end{pmatrix} and\ \mathbf{B} = \begin{pmatrix} -5 & 0 \\ 1 & 6 \end{pmatrix}$$

Then

$$A+B = \begin{pmatrix} 3+-5 & -7+0 \\ -3+1 & 4+6 \end{pmatrix} = \begin{pmatrix} -3 & -7 \\ -2 & 10 \end{pmatrix}$$

Additions of matrices, like equality of matrices is defined only of matrices have same dimension.

Theorem 1.1**:**

Addition of matrices is commutative and associative, that is if A, B and C are matrices having the same dimension then:

A + B = B + A (commutative)

A + (b + C) = (A + B) + C (associative)

Definition 1.3

The product of a scalar K and an mxn matrix A = [aij] mxn is the nn,Xn matrix KA = [kaij] mXn for example:

$$6\begin{pmatrix}-1 & 0 & 7\\ 5 & 2 & -11\end{pmatrix}=\begin{pmatrix}6(-1) & 6(0) & 6(7)\\ 6(5) & 6(2) & 6(-11)\end{pmatrix}=\begin{pmatrix}-6 & 0 & 2\\ 30 & 12 & -6\end{pmatrix}$$

Application of Matrices

Definition 1.4**:**

If A = [aij] mxn is mxn matrix and B = [bjk] nxp an nxp matrix, the product AB is the mxp matrix C = [cik] nxp in which

$$Cik=\sum_{j=1}^{n} aij\,bik$$

Example1: if A = $\begin{pmatrix}a11 & a12 & a13\\ a21 & a22 & a23\end{pmatrix}_{2x3}$ $and\ B=\begin{pmatrix}b11\\ b21\\ b22\end{pmatrix}_{3\times1}$

A B = $\begin{pmatrix}a11\ b11+a12\ b21+a13\ b31\\ a21\ b11+a22\ b21+a23\ b31\end{pmatrix}_{2\times1}$

Example 2: Let A = $\begin{pmatrix}2 & 3\\ -1 & 4\\ 5 & -2\end{pmatrix}_{3\times2}$ $and\ B=\begin{pmatrix}3 & 1 & 4 & -5\\ -2 & 0 & 3 & 4\end{pmatrix}$

A B = $\begin{pmatrix}0 & 2 & 17 & 2\\ -11 & -1 & 8 & 21\\ 19 & 5 & 14 & -33\end{pmatrix}_{3\times4}$

Note 1.1:

1 – in general if A and B are two matrices. Then A B may not be equal of BA. For example A = $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} and\ BA = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\therefore AB \neq BA$

2 – if A B is defined then its not necessary that B A must also be defined. For example. If A is of order (2×3) and B of order (3×1) then clearly A B is define, but B A is not defined.

1.3 Different Types of matrices:

1 – Row Matrix: A matrix which has exactly one row is called row matrix. For example (1, 2, 3, 4) is row matrix

2 – Column Matrix: A matrix which has exactly one column is called a column matrix for example $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ is a column matrix.

3 – Square Matrix**:** A matrix in which the number of row is equal to the number of columns is called a square matrix for example $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a 2×2 square matrix.

A matrix (A) (n×n) A is said to be order   or to be an n-square matrix.

4 - Null or Zero Matrix**:** A matrix each of whose elements is zero is called null matrix or zero matrix, for example $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a (2×3) null matrix.

5 – Diagonal Matrix: the elements aii are called diagonal of a square matrix (a_{11} a_{22} – a_{nn}) constitute its main diagonal A square matrix whose every element other than diagonal elements is zero is called a diagonal matrix for

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6 – Scalar Matrix: A diagonal matrix, whose diagonal elements are equal, is called a scalar matrix.

For example $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are scalar matrix

7 – Identity Matrix: A diagonal matrix whose diagonal elements are all equal to 1 (unity) is called identity matrix or (unit matrix). And denoted by in for

Example $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Note1.2: if A is (mxn) matrix, it is easily to define that AIn = A and also ImA = A

Ex: Find AI and IA when $A = \begin{pmatrix} 3 & 7 & 2 \\ 1 & -1 & 3 \end{pmatrix}$

Solution: IA ⟶ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times2} \begin{pmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{pmatrix}_{2\times3} = \begin{pmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{pmatrix}_{2\times3}$

And AI = $\begin{pmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{pmatrix}_{2\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3} = \begin{pmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{pmatrix}_{2\times3}$

8 – Triangular Matrix**:** A square matrix (aij) whose element aij = 0 whenever $j \langle i$ is Called a lower triangular matrix.simillary y a square matrix (aij) whose element aij = 0 whenever is called an upper Tringular Matrix

For example: $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ are lower triangular matrix

And

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, are upper triangular

Definition 1.4:

Transpose of matrix

The transpose of an mxn matrix A is the nxm matrix denoted by A^T , formed by interchanging the rows and columns of A the ith rows of A is the ith columns in A^T**.**

For Example**:** A = $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}_{2\times3}$ $A^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{pmatrix}_{3\times2}$

9 – Symmetric Matrix: A square matrix A such that A = A^T is called symmetric matrix i.e. A is a symmetric matrix if and only if aij = a_{ji} for all element.

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

For Example: (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, (b)

10 – Skew symmetric Matrix: A square matrix A such that $A = A^T$ is called that A is skew symmetric matrix. i.e A is skew matrix $\longleftrightarrow$ $a_{ji} = -a_{ij}$ for all element of A.

The following are examples of symmetric and skew – symmetric matrices respectively

$$(a)\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix},(b)\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

(a) symmetric

(b) Skew – symmetric.

Note the fact that the main diagonal element of a skew – symmetric matrix must all be Zero

11 – Determinates: To every square matrix that is assigned a specific number called the determinates of the matrix.

(a) Determinates of order one: write det (A) or $|A|$ for detrimental of the matrix A. it is a number assigned to square matrix only.

The determinant of (1×1) matrix (a) is the number a itself det (a) = a.

(c) Determinants of order two: the determinant of the 2×2. matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Is denoted and defined as follows: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Theorem 1.2: determinant of a product of matrices is the product of the determinant of the matrices is the product of the determinant of the matrices det (A B) = det (A). det (B) det (A + B) # det a + det B

(C) Determinates of order three:

(i) the determinant of matrix is defined as follows:

$$\begin{vmatrix} + & - & + \\ a11 & a22 & a23 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix} = a11 \begin{vmatrix} a22 & a23 \\ a32 & a33 \end{vmatrix} - a12 \begin{vmatrix} a21 & a23 \\ a31 & a33 \end{vmatrix} + a13 \begin{vmatrix} a21 & a22 \\ a31 & a32 \end{vmatrix}$$

(ii) Consider the (3×3) matrix $\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$

= a11 a22 a33 + a21 a22 a31 + a13 a21 a32

\- a31 a22 a13 - a32 a23 a11 -a33 a21 a12.

Show that the diagram papering below where the first two columns are rewritten to the right of the matrix.

Theorem 1.3:

A matrix is invertible if and only if its determinant is not Zero usually a matrix is said to be singular if determinant is zero and non singular it otherwise.

1.5 prosperities of Determinants

(1) det A = det A^T where A^T is the transpose of A.

(2) if any two rows (or two columns) of a determinates are interchanged the value of determinants is multiplied by -1.

(3) if all elements in row (or column) of a square matrix are zero.

Then det (A) = 0

(4) if two parallel column (rows) of square matrix A are equal then det (A) = 0

(5) if all the elements of one row (or one column) of a determinant are multiplied by the same factor K. the value of the new determinant is K times the given det.

Example;

$$\begin{pmatrix} 4 & 6 & 1 \\ 3 & -9 & 2 \\ -1 & 12 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 2.3 & 1 \\ 3 & -3.3 & 2 \\ -1 & 4.3 & 3 \end{pmatrix}$$

$$= 3\begin{pmatrix} 4 & 2 & 1 \\ 3 & -3 & 2 \\ -1 & 4 & 3 \\ & & \end{pmatrix}$$

Example: $\begin{pmatrix} 1 & 0 & 4 \\ -2 & 5 & -8 \\ 3 & 6 & 12 \end{pmatrix} = 4\begin{pmatrix} 1 & 0 & 1 \\ -2 & 5 & -2 \\ 3 & 6 & 3 \end{pmatrix} = 0$

(6) if to each element of a selected row (or column) of a square matrix = k times. The corresponding element of another selected row (or column) is added.

Example: $\begin{vmatrix} 2 & 0 & 2 \\ 1 & -1 & +1 \\ 3 & 0 & 2 \end{vmatrix} = -1\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = 2$

$2\times$ row (1) + row (3) $\begin{vmatrix} 2 & 0 & 2 \\ 1 & -1 & 1 \\ 7 & 0 & 6 \end{vmatrix} = -1\begin{vmatrix} 2 & 2 \\ 7 & 6 \end{vmatrix} = 2$

(7) if any row or column contain zero elements and only one element not zero then the determinant will reduced by elementary the row and column if the specified element indeterminate.

1.6 Rank of Matrix: we defined the rank of any matrix a that the order of the largest square sub-matrix of a whose determinant not zero (det of sub-matrix ‡ 0)

Example: Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ find the rank of A

$1 \times 9 \times 5 + 2 \times 6 \times 7 + 3 \times 4 \times 8 - 3 \times 5 \times 7 - 1 \times 6 \times 8 - 2 \times 4 \times 9 = 0$

Since $|A|$ of order 3 Rank ‡ 3

Since $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \neq 0$ the rank ‡ 2

1.7 Minor of matrix: Let A = $\begin{pmatrix} a11 & a12 & a1n \\ a21 & a22 & a2n \\ \\ an1 & an2 & ann \end{pmatrix}$ (4)

Is the square matrix of order n then the determinant of any square sub-matrix of a with order (n-1) obtained by deleting row and column is called the minor of A and denoted by Mij.

1.8 Cofactor of matrix: Let A be square matrix in (4) with mij which is the minors of its. Then the Cofactor of a defined by Cij = $(-1)^{i+j}$ Mij

Example: Let A = $\begin{pmatrix} -2 & 4 & 1 \\ 4 & 5 & 7 \\ -6 & 1 & 0 \end{pmatrix}$ find the minor and the cofactor of element 7.

Solution: The minor of element 7 is

M23 = det $\begin{pmatrix} -2 & 4 \\ -6 & 1 \end{pmatrix} = \begin{vmatrix} -2 & 4 \\ -6 & 1 \end{vmatrix} = 22$

i.e (denoted by take the square sub-matrix by deleting the second rows and third column in A).

the Cofactor of 7 is

$$C23=(-1)^{2+3}M_{23}=(-1)^{2+3}\begin{vmatrix}-2 & 4\\ -6 & 1\end{vmatrix}=-22$$

1.9 Adjoint of matrix: Let matrix A in (4) then the transposed of matrix of cofactor of this matrix is called adjoint of A, adjoint A = transposed matrix of Cofactor.

The inverse of matrix**:** Let A be square matrix. Then inverse of matrix {Where A is non-singular matrix} denoted by A^{-1} and $A^{-1} = \frac{1}{\det A} adj(A)$

1.0 method to find the inverse of A**:** To find the inverse of matrix we must find the following:

(i) the matrix of minor of elements of A.

(ii) the Cofactor of minor of elements of A

(iii) the adjoint of A .

then $A^{-1} = \frac{1}{|A|} adjA$

Example: let $A = \begin{pmatrix}2 & 3 & -4\\ 1 & 2 & 3\\ 3 & -1 & -1\end{pmatrix}$ Find A^{-1}

(1) Minors of A is Mij = $\begin{pmatrix}1 & -10 & -7\\ -7 & 10 & -11\\ 17 & 10 & 1\end{pmatrix}$

(2) Cofactor of A is (-1) Mij = $\begin{pmatrix}1 & 10 & -7\\ 7 & 10 & 11\\ 17 & -10 & 1\end{pmatrix}$

(3) Adj of A = $\begin{pmatrix}1 & 7 & 17\\ 10 & 10 & -10\\ -7 & 11 & 1\end{pmatrix}$.

(4) det = 60

$$A^{-1} = \frac{1}{60}\begin{pmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{pmatrix}.$$

1.11 Properties of Matrix Multiplication:

1 – (KA) B = K (AB) = A (KB)   K is any number

2 – A (BC) = (AB) C

3 – (A + B) C = AC + BC

4 – C (A + B) = CA + CB

5 – AB ‡ BA (in general)

For example: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$A\,B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B\,A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

A B = ‡ B A

6 – A B = 0 but not necessarily A = 0 or B = 0

For Example: $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, B = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}$

$$A\,B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A \neq 0, B \neq 0$$

But

A B = 0

7 - $\begin{pmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix} = C\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8 – A I = I A = A   where I is identity matrix