1-4- <u>Functions</u> : ***Function*** **is any rule that assigns to each element in one set some element from another set :**

$$y=f(x)$$

The inputs make up the ***domain of the function*** **, and the outputs make up** ***the function's range.***

The variable x is called ***independent variable of the function*** **, and the variable y whose value depends on x is called** ***the dependent variable of the function*** **.**

We must keep two restrictions in mind when we define functions :

1. **We never divide by zero .**
2. **We will deal with real – valued functions only.**

<u>Intervals</u> :

- **The** ***open interval*** **is the set of all real numbers that be strictly between two fixed numbers a and b :**

$$(a,b)\equiv a<x<b$$

- **The** ***closed interval*** **is the set of all real numbers that contain both endpoints :**

$$[a,b]\equiv a\le x\le b$$

- ***Half open interval*** **is the set of all real numbers that contain one endpoint but not both :**

$$[a,b)\equiv a\le x<b$$

$$(a,b]\equiv a<x\le b$$

<u>Composition of functions</u> : **suppose that the outputs of a function f can be used as inputs of a function g . We can then hook f and g together to form a new function whose inputs are the inputs of f and whose outputs are the numbers :**

$$(g_o f)(x)=g(f(x))$$

***<u>EX-9</u>*- Find the domain and range of each function :**

$a)\quad y=\sqrt{x+4}\quad , \quad b)\quad y=\dfrac{1}{x-2}$

$c)\quad y=\sqrt{9-x^2}\quad , \quad d)\quad y=\sqrt{2-\sqrt{x}}$

<u>Sol.</u> - $a)\quad x+4\ge 0\Rightarrow\quad x\ge -4\Rightarrow\quad D_x:\forall x\ge -4\quad ,\quad R_y:\forall y\ge 0$

$b)\quad x-2\ne 0\Rightarrow\quad x\ne 2\Rightarrow\quad D_x:\forall x\ne 2$

$y=\dfrac{1}{x-2}\Rightarrow\quad x=\dfrac{1}{y}+2\Rightarrow\quad R_y:\forall y\ne 0$

$c)\quad 9-x^2\ge 0\Rightarrow\quad -3\le x\le 3\Rightarrow\quad D_x:-3\le x\le 3$

$y=\sqrt{9-x^2}\Rightarrow\quad x=\mp\sqrt{9-y^2}$

$\text{since}\quad 9-y^2\ge 0\Rightarrow\quad -3\le y\le 3$

$\text{since}\quad y\ge 0\quad\Rightarrow\quad R_y:0\le y\le 3$

$d)\qquad 2-\sqrt{x}\ge 0 \Rightarrow \qquad 0\le x\le 4 \Rightarrow \quad D_x : 0\le x\le 4$

$$\left.\begin{array}{ll} if & x=0\Rightarrow \quad y=\sqrt{2} \\ if & x=4\Rightarrow \quad y=0 \end{array}\right\} \Rightarrow \qquad R_y : 0\le y\le \sqrt{2}$$

EX-10- Let $f(x)=\dfrac{x}{x-1}$ and $g(x)=1+\dfrac{1}{x}$.

Find $(g_o f)(x)$ and $(f_o g)(x)$.

Sol.-

$$(g_o f)(x)=g(f(x))=g\left(\frac{x}{x-1}\right)=1+\frac{1}{\frac{x}{x-1}}=\frac{2x-1}{x}$$

$$(f_o g)(x)=f(g(x))=f\left(1+\frac{1}{x}\right)=\frac{1+\frac{1}{x}}{1+\frac{1}{x}-1}=x+1$$

EX-11- Let $(g_o f)(x)=x$ and $f(x)=\dfrac{1}{x}$. Find $g(x)$.

Sol.- $(g_o f)(x)=g\left(\dfrac{1}{x}\right)=x\Rightarrow \quad g(x)=\dfrac{1}{x}$

1-5- Limits and continuity :

Limits : The limit of $F(t)$ as t approaches C is the number L if :
Given any radius $\varepsilon>0$ about L there exists a radius $\delta>0$ about C such that for all t , $0<|t-C|<\delta$ implies $|F(t)-L|<\varepsilon$ and we can write it as :

$$\lim_{t\to C} F(t)=L$$

The limit of a function $F(t)$ as $t\to C$ never depend on what happens when $t=C$.

Right hand limit : $\lim_{t\to C^+} F(t)=L$

The limit of the function $F(t)$ as $t\to C$ from the right equals L if :
Given any $\varepsilon>0$ (radius about L) there exists a $\delta>0$ (radius to the right of C) such that for all t :

$$C<t<C+\delta \Rightarrow \quad |F(t)-L|<\varepsilon$$

Left hand limit : $\lim_{t\to C^-} F(t)=L$

The limit of the function $F(t)$ as $t\to C$ from the left equal L if :
Given any $\varepsilon>0$ there exists a $\delta>0$ such that for all t :

$$C-\delta<t<C\Rightarrow \quad |F(t)-L|<\varepsilon$$

Note that **– A function $F(t)$ has a limit at point C if and only if the right hand and the left hand limits at C exist and equal . In symbols :**

$$\lim_{t\to C} F(t) = L \quad \Leftrightarrow \quad \lim_{t\to C^+} F(t) = L \quad and \quad \lim_{t\to C^-} F(t) = L$$

The limit combinations theorems :

1) $\lim\left[F_1(t) \mp F_2(t)\right] = \lim F_1(t) \mp \lim F_2(t)$

2) $\lim\left[F_1(t) * F_2(t)\right] = \lim F_1(t) * \lim F_2(t)$

3) $\lim \dfrac{F_1(t)}{F_2(t)} = \dfrac{\lim F_1(t)}{\lim F_2(t)} \quad where \quad \lim F_2(t) \neq 0$

4) $\lim\left[k * F_1(t)\right] = k * \lim F_1(t) \quad \forall k$

5) $\lim\limits_{\theta\to 0} \dfrac{Sin\theta}{\theta} = 1$

provided that θ is measured in radius

The limits (in $1 - 4$) are all to be taken as $t \to C$ and $F_1(t)$ and $F_2(t)$ are to be real functions .

<u>Thm. -1 : The sandwich theorem</u> : **Suppose that $f(t) \le g(t) \le h(t)$ for all $t \neq C$ in some interval about C and that $f(t)$ and $h(t)$ approaches the same limit L as $t \to C$, then :**

$$\lim_{t\to C} g(t) = L$$

<u>Infinity as a limit</u> :

1.The limit of the function $f(x)$ as x approaches infinity is the number L:
$\lim\limits_{x\to\infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number M such that for all x : $M < x \Rightarrow |f(x) - L| < \varepsilon$.

2. The limit of $f(x)$ as x approaches negative infinity is the number L :
$\lim\limits_{x\to -\infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number N such that for all x : $x < N \Rightarrow |f(x) - L| < \varepsilon$.

The following facts are some times abbreviated by saying :

a) As x approaches 0 from the right , $1/x$ tends to ∞ .
b) As x approaches 0 from the left , $1/x$ tends to $-\infty$.
c) As x tends to ∞ , $1/x$ approaches 0 .
d) As x tends to $-\infty$, $1/x$ approaches 0 .

<u>Continuity</u> :

<u>Continuity at an interior point</u> : **A function $y = f(x)$ is continuous at an interior point C of its domain if : $\lim\limits_{x\to C} f(x) = f(C)$.**

<u>Continuity at an endpoint</u> : **A function $y = f(x)$ is continuous at a left endpoint a of its domain if : $\lim\limits_{x\to a^+} f(x) = f(a)$.**

A function $y = f(x)$ is continuous at a right endpoint b of its domain if: $\lim\limits_{t\to b^-} f(x) = f(b)$.

Continuous function **: A function is continuous if it is continuous at each point of its domain .**

Discontinuity at a point **: If a function f is not continuous at a point C, we say that f is discontinuous at C, and call C a point of discontinuity of f .**

The continuity test **: The function $y = f(x)$ is continuous at $x = C$ if and only if all three of the following statements are true :**

1) $f(C)$ exist (C is in the domain of f) .
2) $\lim_{x \to C} f(x)$ exists (f has a limit as $x \to C$) .
3) $\lim_{x \to C} f(x) = f(C)$ (the limit equals the function value) .

Thm.-2 : The limit combination theorem for continuous function :

If the function f and g are continuous at $x = C$, then all of the following combinations are continuous at $x = C$:

$$1)\ f \mp g \quad 2)\ f.g \quad 3)\ k.g \ \ \forall k \quad 4)\ g_o f\ ,\ f_o g \quad 5)\ f/g$$

provided $g(C) \neq 0$

Thm.-3 **: A function is continuous at every point at which it has a derivative . That is , if $y = f(x)$ has a derivative $f'(C)$ at $x = C$, then f is continuous at $x = C$.**

EX-12 **– Find :**

$$1)\ \lim_{x \to 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} \quad , \quad 2)\ \lim_{x \to a} \frac{x^3 - a^3}{x^4 - a^4}$$

$$3)\ \lim_{x \to 0} \frac{Sin 5x}{Sin 3x} \quad , \quad 4)\ \lim_{y \to 0} \frac{tan 2y}{3y}$$

$$5)\ \lim_{x \to 0} \frac{Sin 2x}{2x^2 + x} \quad , \quad 6)\ \lim_{x \to \infty} \left(1 + Cos \frac{1}{x} \right)$$

$$7)\ \lim_{x \to \infty} \frac{3x^3 + 5x^2 - 7}{10x^3 - 11x + 5} \quad , \quad 8)\ \lim_{y \to \infty} \frac{3y + 7}{y^2 - 2}$$

$$9)\ \lim_{x \to \infty} \frac{x^3 - 1}{2x^2 - 7x + 5} \quad , \quad 10)\ \lim_{x \to -1^-} \frac{1}{x + 1}$$

$$11)\ \lim_{x \to 0} Cos\left(1 - \frac{Sinx}{x} \right) \quad , \quad 12)\ \lim_{x \to 0} Sin\left(\frac{\pi}{2} Cos(tan x) \right)$$

S0l.-

$$1)\ \lim_{x \to 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} = \lim_{x \to 0} \frac{5x + 8}{3x^2 - 16} = \frac{0 + 8}{0 - 16} = -\frac{1}{2}$$

$$2)\ \lim_{x \to a} \frac{x^3 - a^3}{x^4 - a^4} = \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2)}{(x - a)(x + a)(x^2 + a^2)} = \frac{a^2 + a^2 + a^2}{(a + a)(a^2 + a^2)} = \frac{3}{4a}$$

$$3)\ \lim_{x \to 0} \frac{5\frac{Sin 5x}{5x}}{3\frac{Sin 3x}{3x}} = \frac{5}{3} . \frac{\lim_{5x \to 0} \frac{Sin 5x}{5x}}{\lim_{3x \to 0} \frac{Sin 3x}{3x}} = \frac{5}{3}$$

$$4)\ \lim_{y\to 0}\frac{\tan 2y}{3y}=\frac{2}{3}.\lim_{2y\to 0}\frac{Sin2y}{2y}.\lim_{y\to 0}\frac{1}{Cos2y}=\frac{2}{3}$$

$$5)\ \lim_{x\to 0}\frac{Sin2x}{2x^2+x}=2\lim_{2x\to 0}\frac{Sin2x}{2x}.\lim_{x\to 0}\frac{1}{2x+1}=2$$

$$6)\ \lim_{x\to\infty}\left(1+Cos\frac{1}{x}\right)=1+Cos0=2$$

$$7)\ \lim_{x\to\infty}\frac{3x^3+5x^2-7}{10x^3-11x+5}=\lim_{x\to\infty}\frac{3+\frac{5}{x}-\frac{7}{x^3}}{10-\frac{11}{x^2}+\frac{5}{x^3}}=\frac{3}{10}$$

$$8)\ \lim_{y\to\infty}\frac{3y+7}{y^2-2}=\lim_{y\to\infty}\frac{\frac{3}{y}+\frac{7}{y^2}}{1-\frac{2}{y^2}}=\frac{0}{1}=0$$

$$9)\ \lim_{x\to\infty}\frac{x^3-1}{2x^2-7x+5}=\lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{\frac{2}{x}-\frac{7}{x^2}+\frac{5}{x^3}}=\frac{1}{0}=\infty$$

$$10)\ \lim_{x\to -1^-}\frac{1}{x+1}=\frac{1}{-1+1}=-\infty$$

$$11)\ \lim_{x\to 0}Cos\left(1-\frac{Sinx}{x}\right)=Cos\left(1-\lim_{x\to 0}\frac{Sinx}{x}\right)=Cos0=1$$

$$12)\ \lim_{x\to 0}Sin\left(\frac{\pi}{2}Cos(\tan x)\right)=Sin\left(\frac{\pi}{2}Cos(\tan 0)\right)=Sin\left(\frac{\pi}{2}Cos0\right)=Sin\frac{\pi}{2}=1$$

EX-13- **Test continuity for the following function :**

$$f(x)=\begin{cases} x^2-1 & -1\le x<0 \\ 2x & 0\le x<1 \\ 1 & x=1 \\ -2x+4 & 1<x\le 2 \\ 0 & 2<x\le 3 \end{cases}$$

Sol.- **We test the continuity at midpoints** $x=0\ ,\ 1\ ,\ 2$ **and endpoints** $x=-1\ ,\ 3$ **.**

$At\quad x=0\Rightarrow\qquad f(0)=2*0=0$

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0}(x^2-1)=-1$$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0}2x=0\neq\lim_{x\to 0^-}f(x)$$

$$Since\quad \lim_{x\to 0}f(x)\ doesn't\ exist$$

$$Hence\quad the\ function\ discontinuous\ at\ x=0$$

$At \quad x=1 \Rightarrow \quad f(1)=1$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1} 2x=2$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(-2x+4)=2=\lim_{x\to 1^-} f(x)=\lim_{x\to 1} f(x)$$

$$Since \quad \lim_{x\to 1} f(x)\neq f(1)$$

Hence the function is discontinuous at $x=1$

$At \quad x=2 \Rightarrow \quad f(2)=-2*2+4=0$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2}(-2x+4)=0$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2} 0=0=\lim_{x\to 2^-} f(x)=\lim_{x\to 2} f(x)$$

$$Since \quad \lim_{x\to 2} f(x)=f(2)=0$$

Hence the function is continuous at $x=2$

$At \quad x=-1 \Rightarrow \quad f(-1)=(-1)^2-1=0$

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1}(x^2-1)=0=f(-1)$$

Hence the function is continuous at $x=-1$

$At \quad x=3 \Rightarrow \quad f(3)=0$

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3} 0=0=f(3)$$

Hence the function is continuous at $x=3$

<u>*EX-14*</u>- What value should be assigned to a **to make the function :**

$$f(x)=\begin{cases} x^2-1 & x<3 \\ 2ax & x\geq 3 \end{cases}$$ **continuous at** $x=3$ **?**

<u>*Sol.*</u> –

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)\Rightarrow \lim_{x\to 3}(x^2-1)=\lim_{x\to 3} 2ax \Rightarrow \quad 8=6a \Rightarrow \quad a=\frac{4}{3}$$