Al-Mustaqbal University
College of Sciences
Intelligent Medical Systems Department

جامـــعـة الـمـــسـتـقـبـل
AL MUSTAQBAL UNIVERSITY

# كلية العـــــلوم
# قســـــم علوم الحياة

## LECTURE: (2)

**Subject:** **Functions**
**Level: First**
**Lecturer: Dr. Mustafa Talal**

# Chapter two
# Functions

## 2-1- Exponential and Logarithm functions :

***Exponential functions :*** **If $a$ is a positive number and $x$ is any number , we define the exponential function as :**

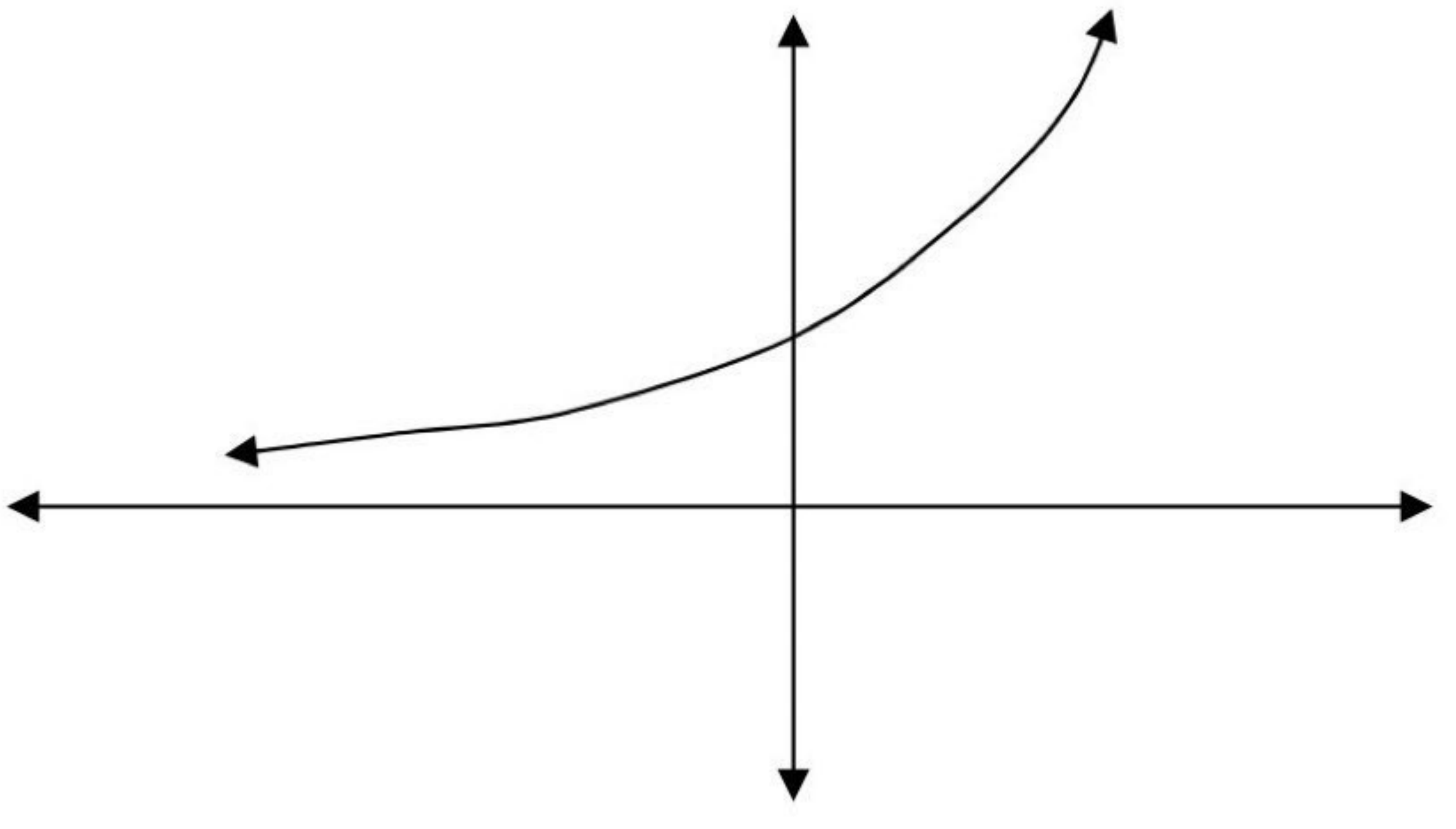
$y = a^x$ **with domain :** $-\infty < x < \infty$

**Range :** $y > 0$

**The properties of the exponential functions are :**

1. **If** $a > 0 \leftrightarrow a^x > 0$ .
2. $a^x . a^y = a^{x+y}$ .
3. $a^x / a^y = a^{x-y}$ .
4. $(a^x)^y = a^{x.y}$ .
5. $(a . b)^x = a^x . b^x$ .
6. $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$ .
7. $a^{-x} = 1 / a^x$ *and* $a^x = 1 / a^{-x}$ .
8. $a^x = a^y \leftrightarrow x = y$ .
9. $a^0 = 1$ ,
   $a^{\infty} = \infty$ , $a^{-\infty} = 0$ , *where* $a > 1$ .
   $a^{\infty} = 0$ , $a^{-\infty} = \infty$ , *where* $a < 1$ .

**The graph of the exponential function** $y = a^x$ **is :**



***Logarithm function :*** **If $a$ is any positive number other than $1$ , then the logarithm of $x$ to the base $a$ denoted by :**

$y = \log_a x$ *where* $x > 0$

**At** $a = e = 2.7182828\ldots$ **, we get the natural logarithm and denoted by :**

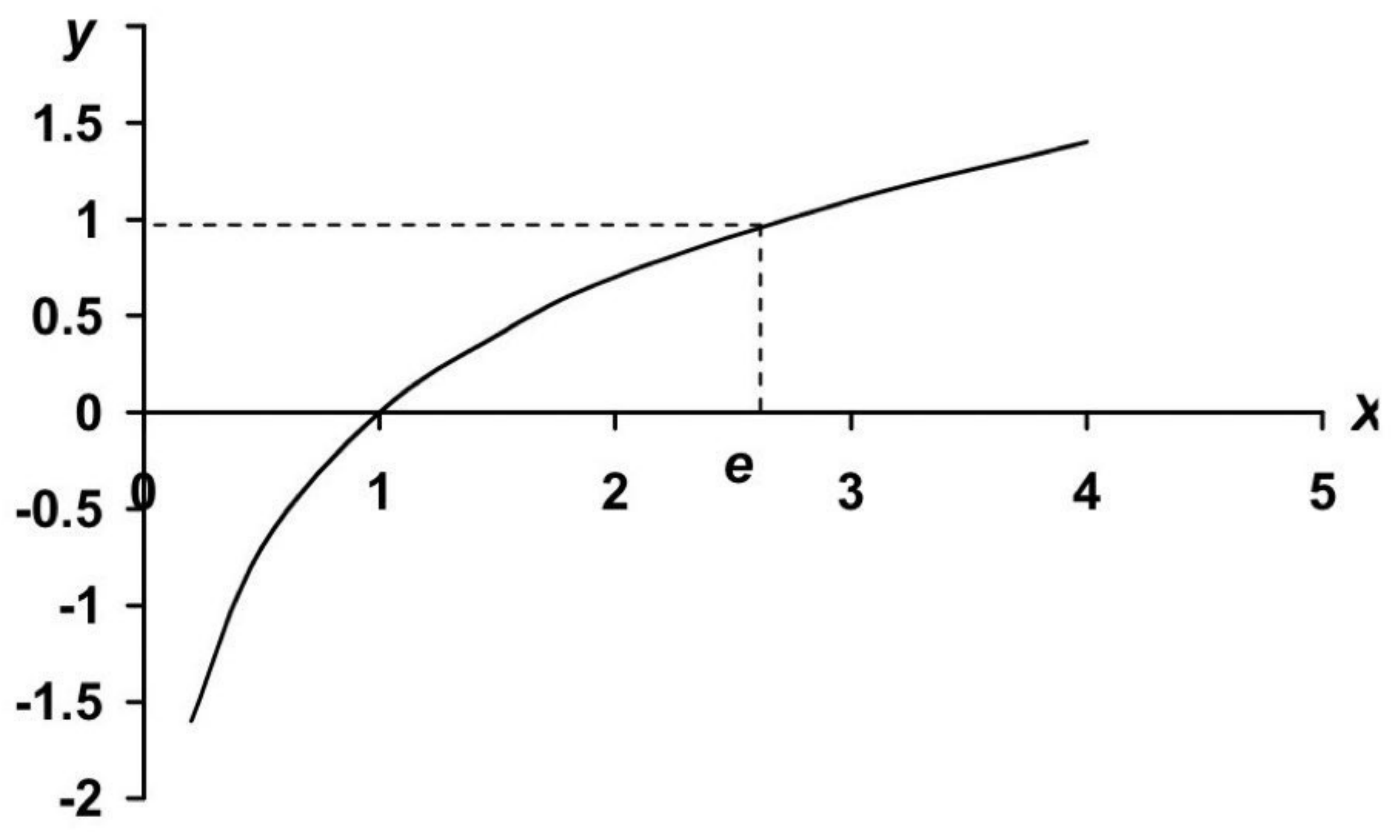
$y = \ln x$

**Let** $x , y > 0$ **then the properties of logarithm functions are :**

1. $y = a^x \leftrightarrow x = \log_a y$ *and* $y = e^x \leftrightarrow x = \ln y$ .
2. $\log_e x = \ln x$ .
3. $\log_a x = \ln x / \ln a$ .

4. $ln\,(x.y) = ln\,x + ln\,y$ .
5. $ln\,(x/y) = ln\,x - ln\,y$ .
6. $ln\,x^n = n.\,ln\,x$ .
7. $ln\,e = log_a a = 1$ and $ln\,1 = log_a 1 = 0$ .
8. $a^x = e^{x.\,ln\,a}$ .
9. $e^{ln\,x} = x$ .

**The graph of the function $y = ln\,x$ is :**




***Application of exponential and logarithm functions*** **:**

**We take Newton's law of cooling :**

$$T - T_S = (T_0 - T_S)\,e^{t\,k}$$

**where $T$ is the temperature of the object at time $t$.**

**$T_S$ is the surrounding temperature .**

**$T_0$ is the initial temperature of the object .**

**$k$ is a constant .**

***EX-1-*** **The temperature of an ingot of metal is $80\ ^oC$ and the room temperature is $20\ ^oC$ . After twenty minutes, it was $70\ ^oC$ .**

**a) What is the temperature will the metal be after $30$ minutes?**
**b) What is the temperature will the metal be after two hours?**
**c) When will the metal be $30\ ^oC$?**

***Sol.*** **:**

$$T - T_S = (T_0 - T_S)e^{tk} \Rightarrow 50 = 60\,e^{20\,k} \Rightarrow k = \frac{ln\,5 - ln\,6}{20} = -0.0091$$

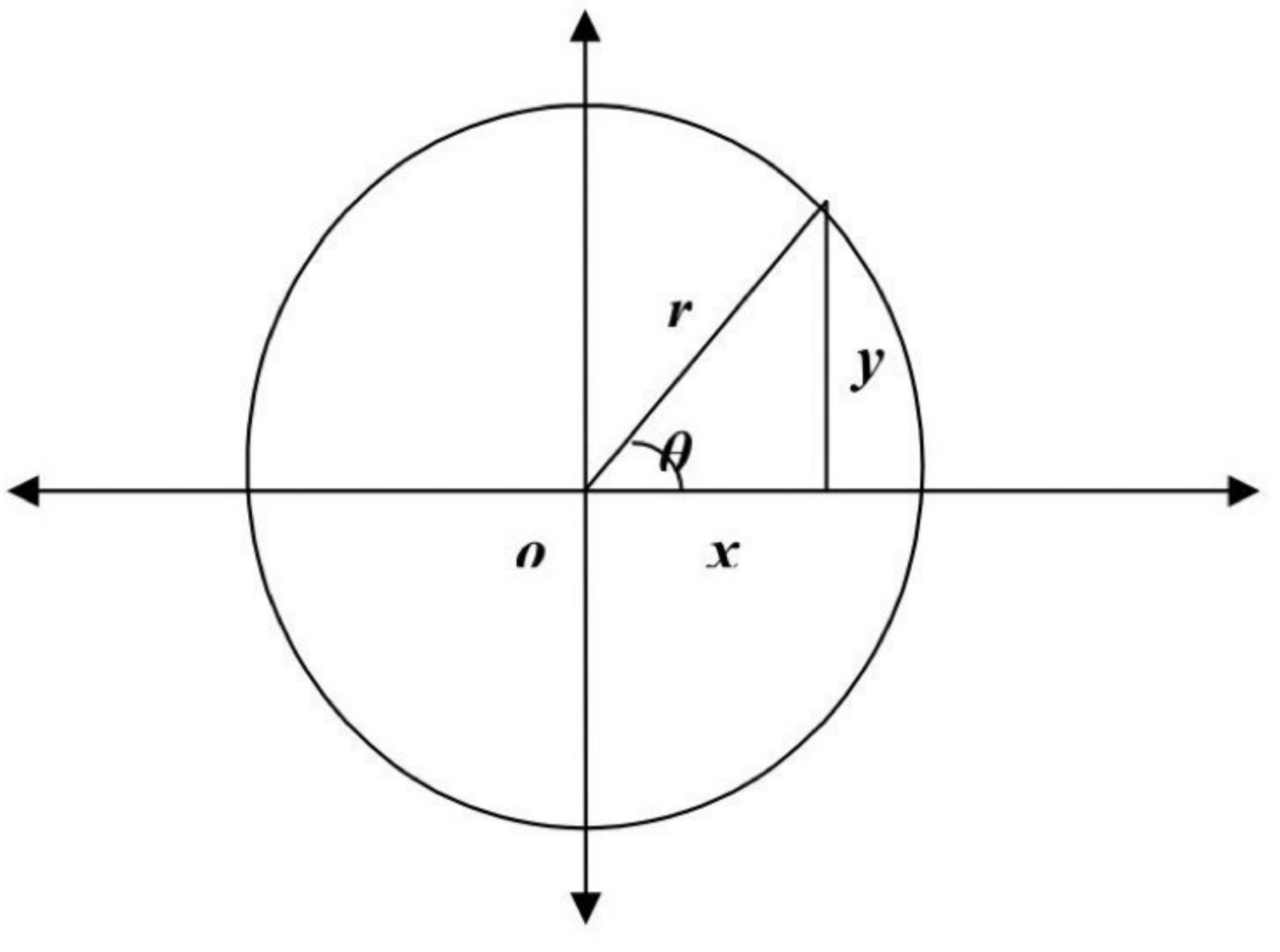
$$a)\quad T - 20 = 60e^{30(-0.0091)} = 60 * 0.761 = 45.6\ ^oC \Rightarrow T = 65.6\ ^oC$$

$$b)\quad T - T_S = 60e^{120(-0.0091)} = 60 * 0.335 = 20.1\ ^oC \Rightarrow T = 40.1\ ^oC$$

$$c)\quad 10 = 60e^{-0.0091\,t} \Rightarrow -0.0091\,t = -\,ln\,6 \Rightarrow t = 3.3\ hrs.$$

**2-2- <u>Trigonometric functions</u> : When an angle of measure $\theta$ is placed in standard position at the center of a circle of radius $r$ , the trigonometric functions of $\theta$ are defined by the equations :**

$$Sin\theta = \frac{y}{r} = \frac{1}{csc\,\theta} \quad , \quad Cos\theta = \frac{x}{r} = \frac{1}{sec\,\theta} \quad , \quad tan\,\theta = \frac{y}{x} = \frac{1}{Cot\theta} = \frac{Sin\theta}{Cos\theta}$$




**The following are some properties of these functions :**

1) $Sin^2\theta + Cos^2\theta = 1$

2) $1 + tan^2\theta = sec^2\theta$ and $1 + Cot^2\theta = csc^2\theta$

3) $Sin(\theta \mp \beta) = Sin\theta.Cos\beta \mp Cos\theta.Sin\beta$

4) $Cos(\theta \mp \beta) = Cos\theta.Cos\beta \pm Sin\theta.Sin\beta$

5) $tan(\theta \mp \beta) = \dfrac{tan\,\theta \mp tan\,\beta}{1 \pm tan\,\theta.tan\,\beta}$

6) $Sin2\theta = 2Sin\theta.Cos\theta$ and $Cos2\theta = Cos^2\theta - Sin^2\theta$

7) $Cos^2\theta = \dfrac{1 + Cos2\theta}{2}$ and $Sin^2\theta = \dfrac{1 - Cos2\theta}{2}$

8) $Sin(\theta \mp \frac{\pi}{2}) = \mp Cos\theta$ and $Cos(\theta \mp \frac{\pi}{2}) = \pm Sin\theta$

9) $Sin(-\theta) = -Sin\theta$ and $Cos(-\theta) = Cos\theta$ and $tan(-\theta) = -tan\,\theta$

10) $Sin\theta.Sin\beta = \frac{1}{2}[Cos(\theta - \beta) - Cos(\theta + \beta)]$

$Cos\theta.Cos\beta = \frac{1}{2}[Cos(\theta - \beta) + Cos(\theta + \beta)]$

$Sin\theta.Cos\beta = \frac{1}{2}[Sin(\theta - \beta) + Sin(\theta + \beta)]$

11) $$Sin\theta + Sin\beta = 2Sin\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

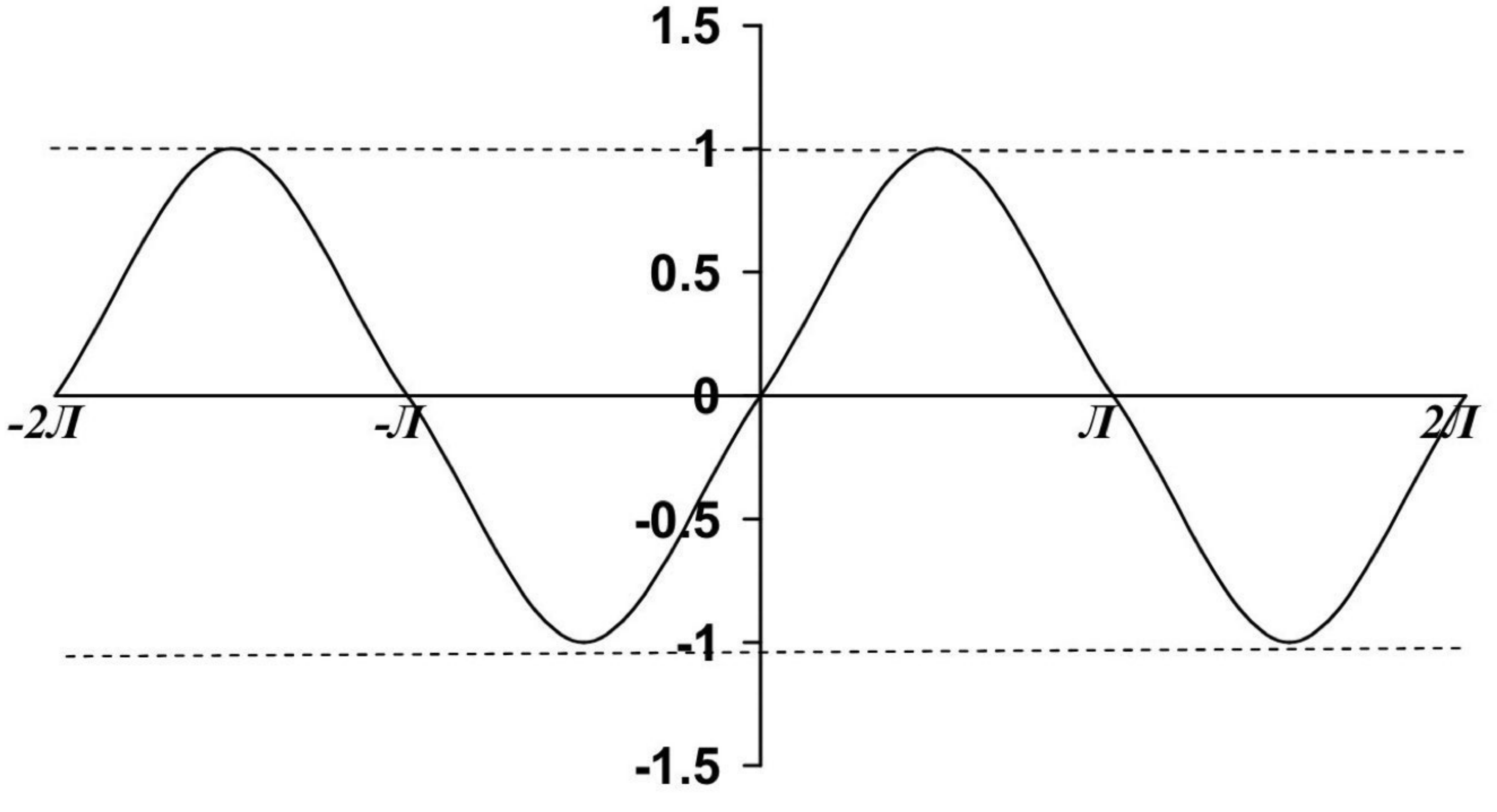
$$Sin\theta - Sin\beta = 2Cos\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

12) $$Cos\theta + Cos\beta = 2Cos\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

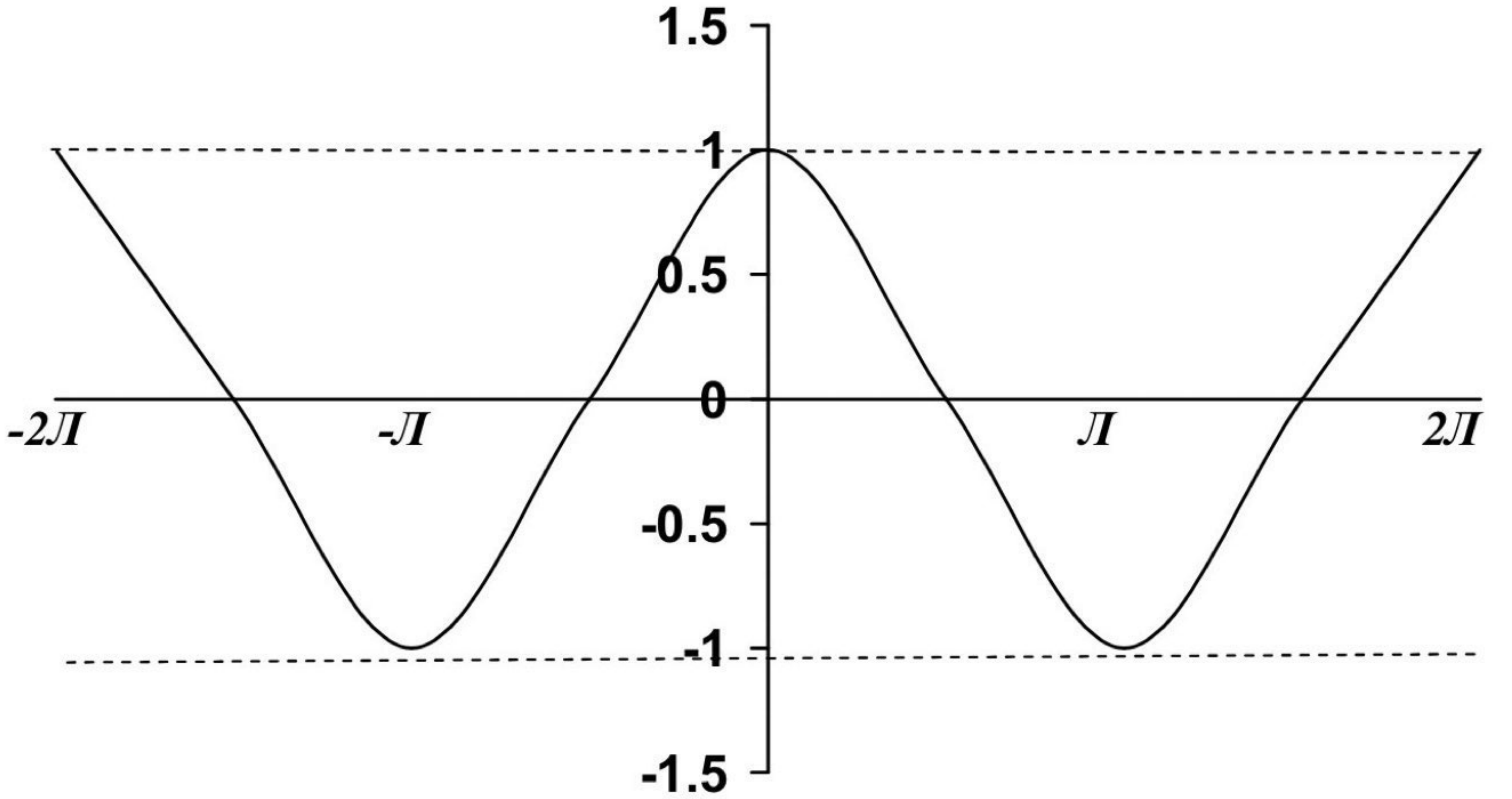
$$Cos\theta - Cos\beta = -2Sin\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

| $\theta$ | 0 | Π / 6 | Π / 4 | Π / 3 | Π / 2 | Π |
|---|---|---|---|---|---|---|
| $Sin\theta$ | 0 | 1/2 | $1/\sqrt{2}$ | $\sqrt{3}/2$ | 1 | 0 |
| $Cos\theta$ | 1 | $\sqrt{3}/2$ | $1/\sqrt{2}$ | 1/2 | 0 | -1 |
| $tan\theta$ | 0 | $1/\sqrt{2}$ | 1 | $\sqrt{3}$ | ∞ | 0 |

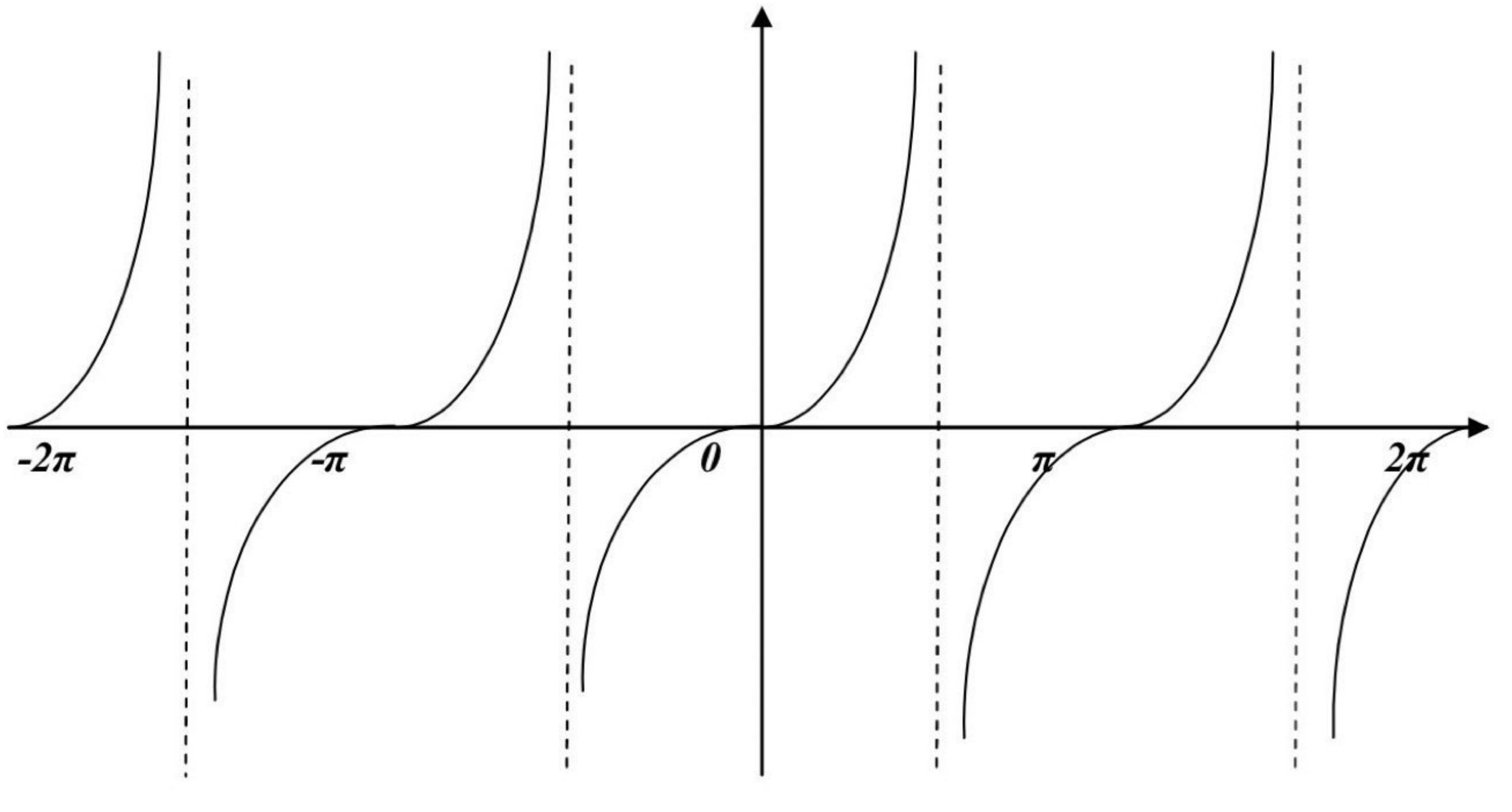
**Graphs of the trigonometric functions are :**



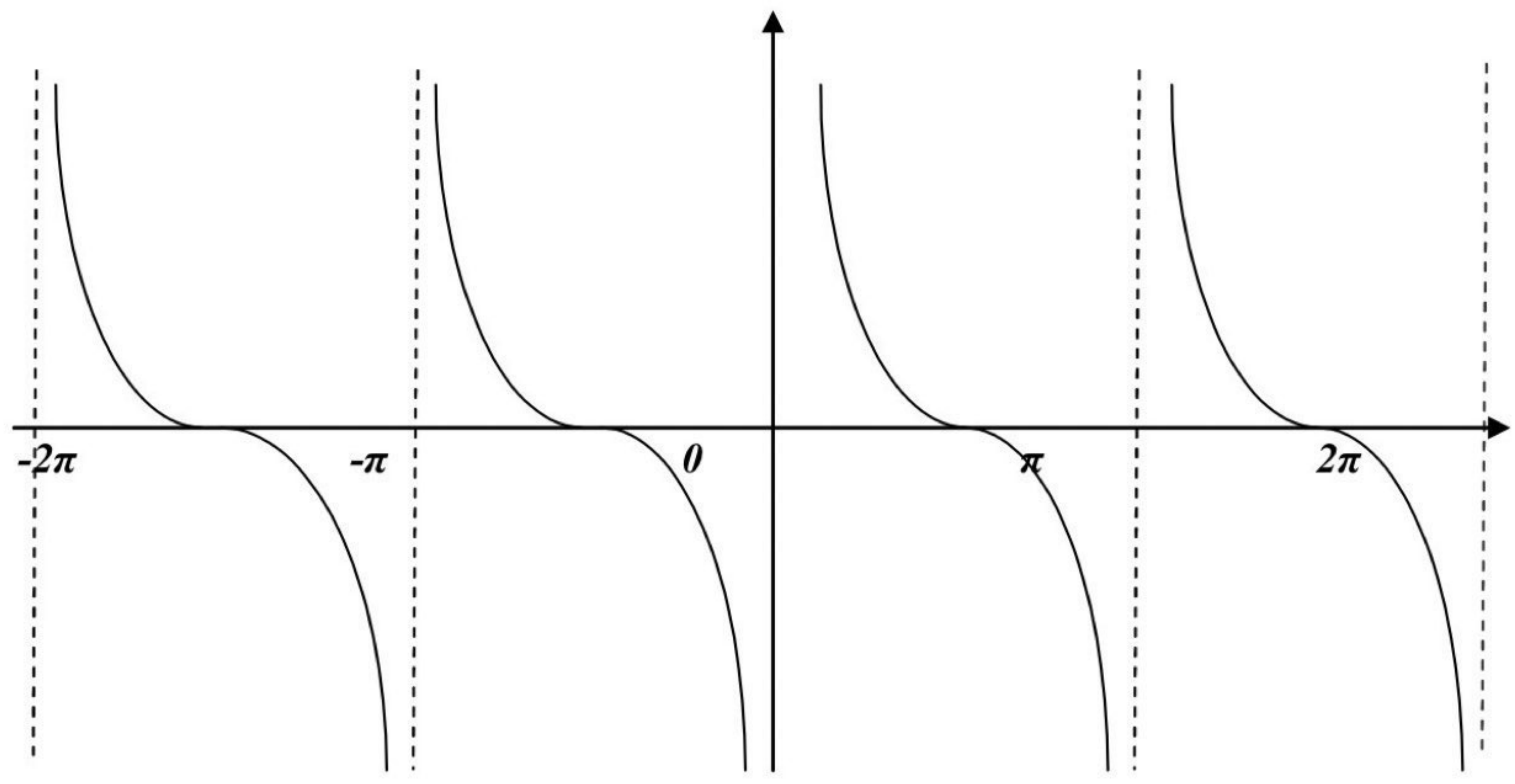

$$y = Sinx \quad D_x : \forall x$$
$$R_y : -1 \le y \le 1$$



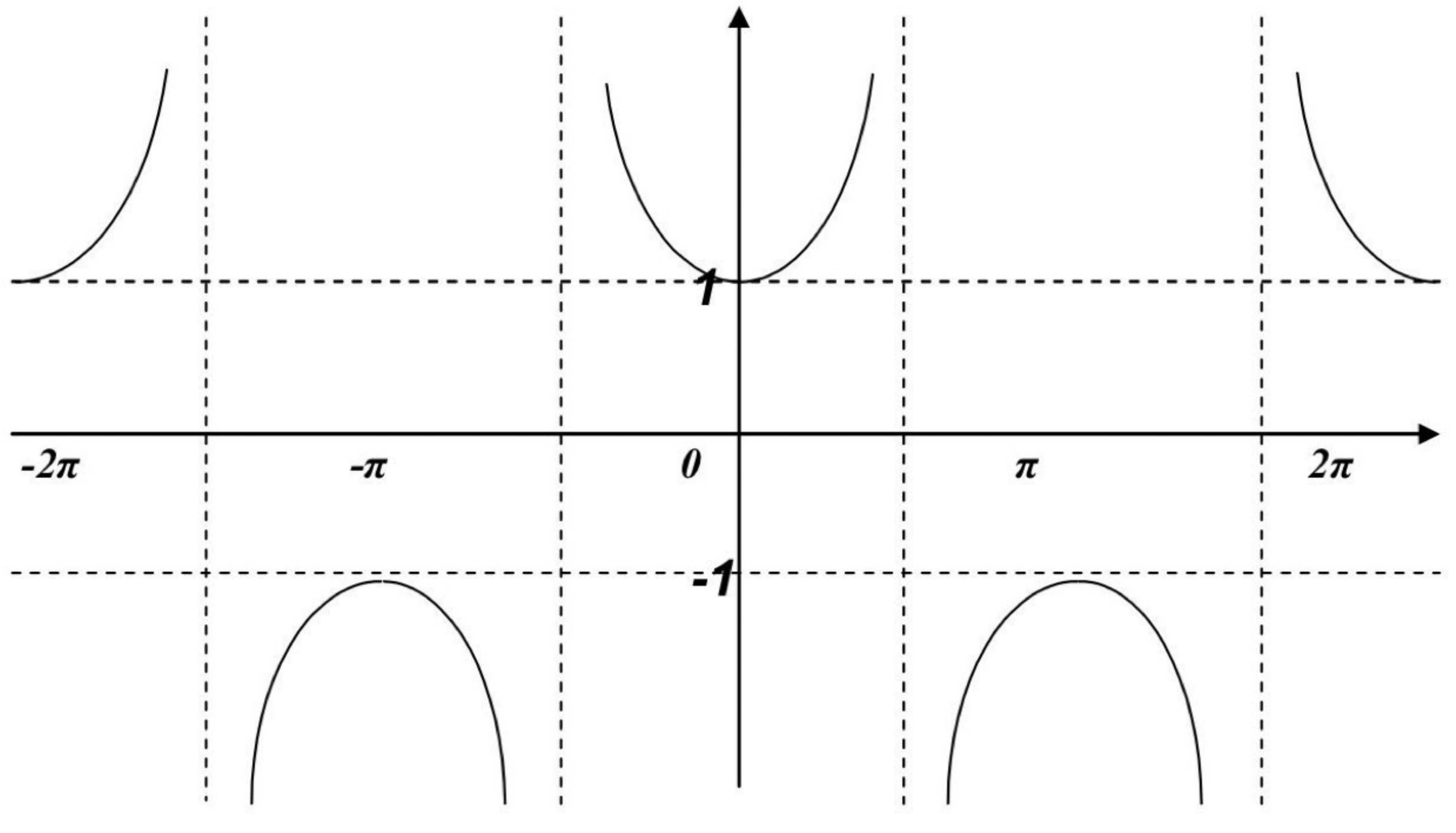

$$y = Cosx \quad D_x : \forall x$$
$$R_y : -1 \le y \le 1$$



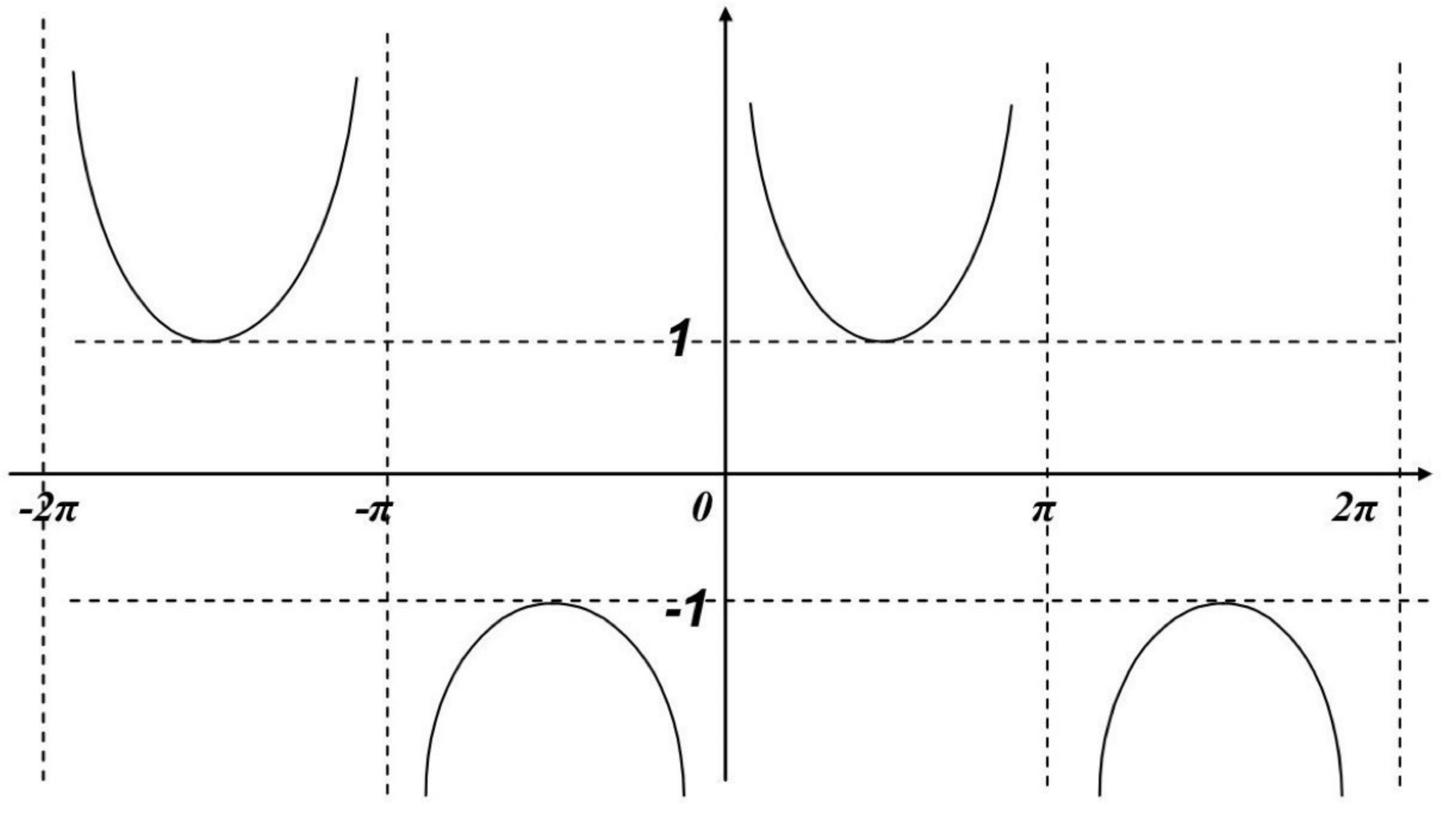

$$y = tan\,x \quad D_x : \forall x \ne \frac{2n+1}{2}\pi$$
$$R_y : \forall y$$




$$y = Cotx \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y$$




$$y = Secx \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$
$$R_y : \forall y \geq 1 \ \ or \ \ y \leq -1$$




$$y = csc\, x \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y \geq 1 \ \text{or} \ y \leq -1$$

*Where* $n = 0, \mp 1, \mp 2, \mp 3, \ldots\ldots$

**<u>EX-2</u> - Solve the following equations , for values of $\theta$ from $0^o$ to $360^o$ inclusive .**

**a)** $tan\ \theta = 2\ Sin\ \theta$ **b)** $1 + Cos\ \theta = 2\ Sin^2\ \theta$

***<u>Sol.</u>-***

$$a)\quad tan\theta = 2Sin\theta \Rightarrow \frac{Sin\theta}{Cos\theta} = 2Sin\theta$$
$$\Rightarrow Sin\theta(1 - 2Cos\theta) = 0$$
$$either\ \ Sin\theta = 0 \Rightarrow \theta = 0^o, 180^o, 360^o$$
$$or\quad Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

**Therefore the required values of $\theta$ are $0^o, 60^o, 180^o, 300^o, 360^o$ .**

$$b)\quad 1 + Cos\theta = 2.Sin^2\theta \Rightarrow 1 + Cos\theta = 2(1 - Cos^2\theta)$$
$$\Rightarrow (2Cos\theta - 1)(Cos\theta + 1) = 0$$
$$either\qquad Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$
$$or\qquad Cos\theta = -1 \Rightarrow \theta = 180^o$$

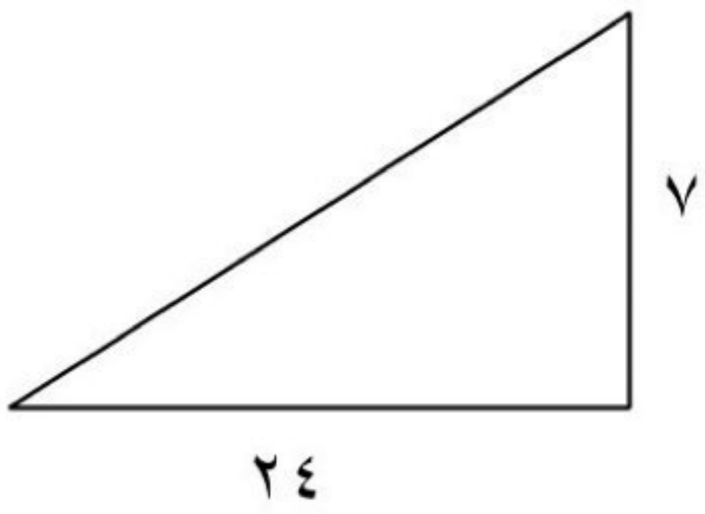
**There the roots of the equation between $0^o$ and $360^o$ are $60^o, 180^o$ and $300^o$ .**

***EX-3*- If $\tan\theta = 7/24$, find without using tables the values of $Sec\theta$ and $Sin\theta$.**

***Sol*.-**

$$\tan\theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{7^2 + 24^2} = 25$$

$$Sec\theta = \frac{r}{x} = \frac{25}{24} \quad and \quad Sin\theta = \frac{y}{r} = \frac{7}{25}$$




***EX-4*- Prove the following identities :**

$$a)\quad Csc\theta + \tan\theta . Sec\theta = Csc\theta . Sec^2\theta$$

$$b)\quad Cos^4\theta - Sin^4\theta = Cos^2\theta - Sin^2\theta$$

$$c)\quad \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta}$$

***Sol.*-**

$$a)\quad L.H.S. = Csc\theta + \tan\theta . Sec\theta = \frac{1}{Sin\theta} + \frac{Sin\theta}{Cos\theta} . \frac{1}{Cos\theta}$$

$$= \frac{Cos^2\theta + Sin^2\theta}{Sin\theta . Cos^2\theta} = \frac{1}{Sin\theta} . \frac{1}{Cos^2\theta} = Csc\theta . Sec^2\theta = R.H.S.$$

$$b)\quad L.H.S. = Cos^4\theta - Sin^4\theta = (Cos^2\theta - Sin^2\theta).(Cos^2\theta + Sin^2\theta)$$

$$= Cos^2\theta - Sin^2\theta = R.H.S.$$

$$c)\quad L.H.S. = \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\dfrac{1}{Cos\theta} - \dfrac{1}{Sin\theta}}{\dfrac{Sin\theta}{Cos\theta} - \dfrac{Cos\theta}{Sin\theta}} = \frac{1}{Sin\theta + Cos\theta}$$

$$= \frac{Sin^2\theta + Cos^2\theta}{Sin\theta + Cos\theta} . \frac{\dfrac{1}{Sin\theta . Cos\theta}}{\dfrac{1}{Sin\theta . Cos\theta}} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta} = R.H.S.$$

***EX-5*- Simplify** $\dfrac{1}{\sqrt{x^2 - a^2}}$ *when* $x = a . Csc\theta$ .

***Sol.*-** $\dfrac{1}{\sqrt{x^2 - a^2}} = \dfrac{1}{\sqrt{a^2 Csc^2\theta - a^2}} = \dfrac{1}{a\sqrt{Cot^2\theta}} = \dfrac{1}{a}\tan\theta$ .

***EX-6*- Eliminate $\theta$ from the equations :**

*i)* $x = a\, Sin\theta$ *and* $y = b\, \tan\theta$

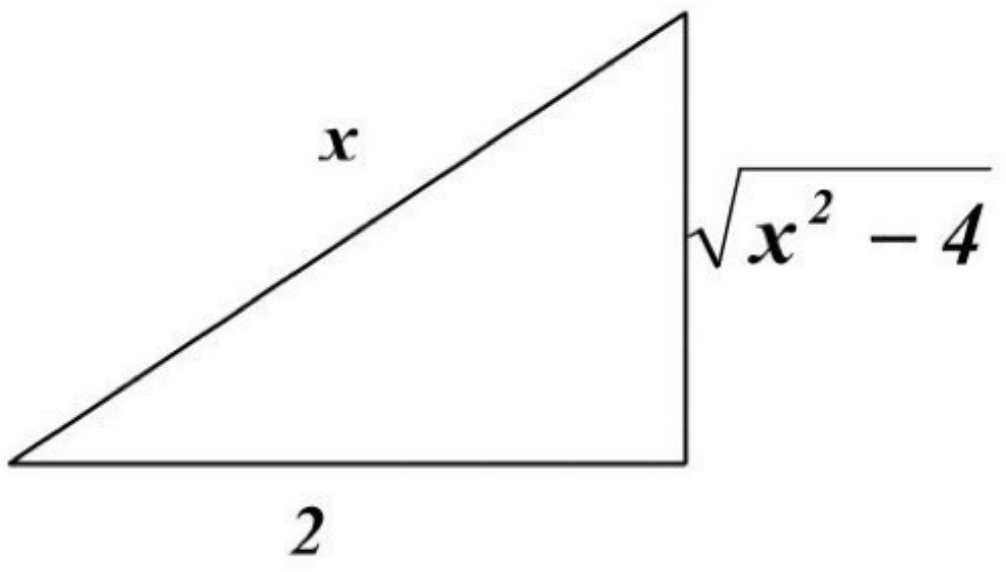
*ii)* $x = 2\, Sec\theta$ *and* $y = Cos2\theta$

***Sol.*-**

i) $x = a.Sin\theta \Rightarrow Sin\theta = \frac{x}{a} \Rightarrow Csc\theta = \frac{a}{x}$

$y = b\tan\theta \Rightarrow \tan\theta = \frac{y}{b} \Rightarrow Cot\theta = \frac{b}{y}$

Since $Csc^2\theta = Cot^2\theta + 1 \Rightarrow \frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$




ii) $x = 2Sec\theta \Rightarrow Cos\theta = \frac{2}{x}$

$y = Cos\,2\theta \Rightarrow y = Cos^2\theta - Sin^2\theta$

$y = \frac{4}{x^2} - \frac{x^2 - 4}{x^2} \Rightarrow x^2 y = 8 - x^2$

<u>EX-7</u>- If $\tan^2\theta - 2\tan^2\beta = 1$ , show that $2\,Cos^2\theta - Cos^2\beta = 0$ .

<u>Sol.</u> –

$\tan^2\theta - 2\tan^2\beta = 1 \Rightarrow Sec^2\theta - 1 - 2(Sec^2\beta - 1) = 1$

$\Rightarrow Sec^2\theta - 2Sec^2\beta = 0 \Rightarrow \frac{1}{Cos^2\theta} - \frac{2}{Cos^2\beta} = 0$

$\Rightarrow 2Cos^2\theta - Cos^2\beta = 0$ Q.E.D.

<u>EX-8</u>- If $a\,Sin\theta = p - b\,Cos\theta$ and $b\,Sin\theta = q + a\,Cos\theta$ .Show that :
$a^2 + b^2 = p^2 + q^2$

<u>Sol.</u>-

$p = a.Sin\theta + b.Cos\theta$ and $q = b.Sin\theta - a.Cos\theta$

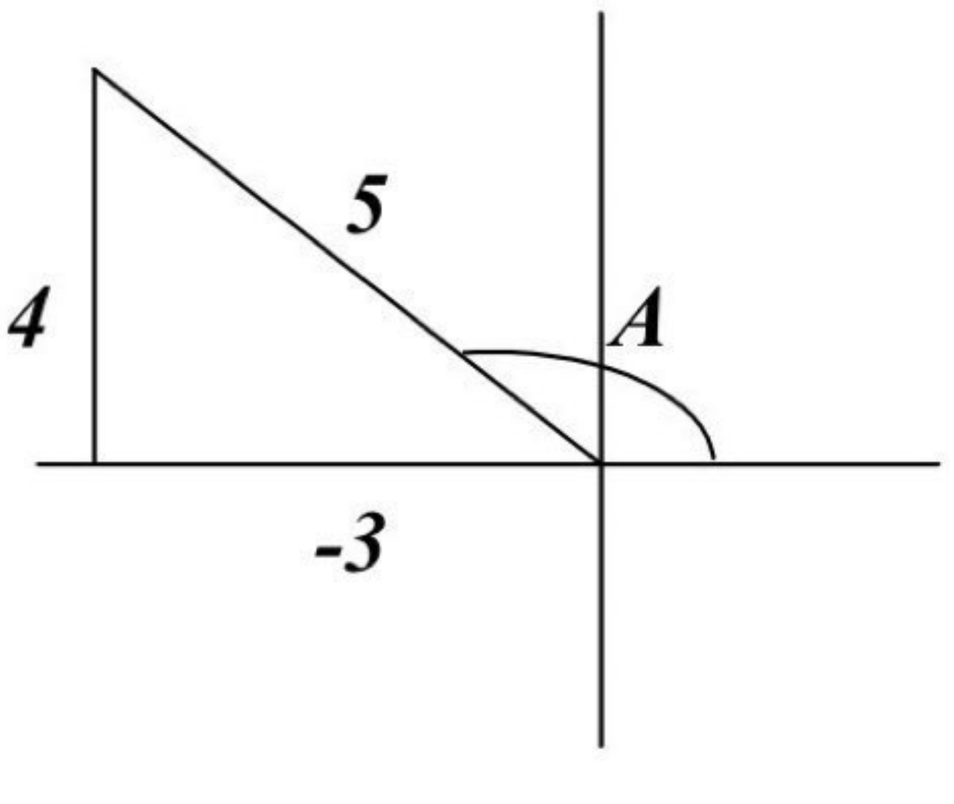
$p^2 + q^2 = (aSin\theta + bCos\theta)^2 + (bSin\theta - aCos\theta)^2$

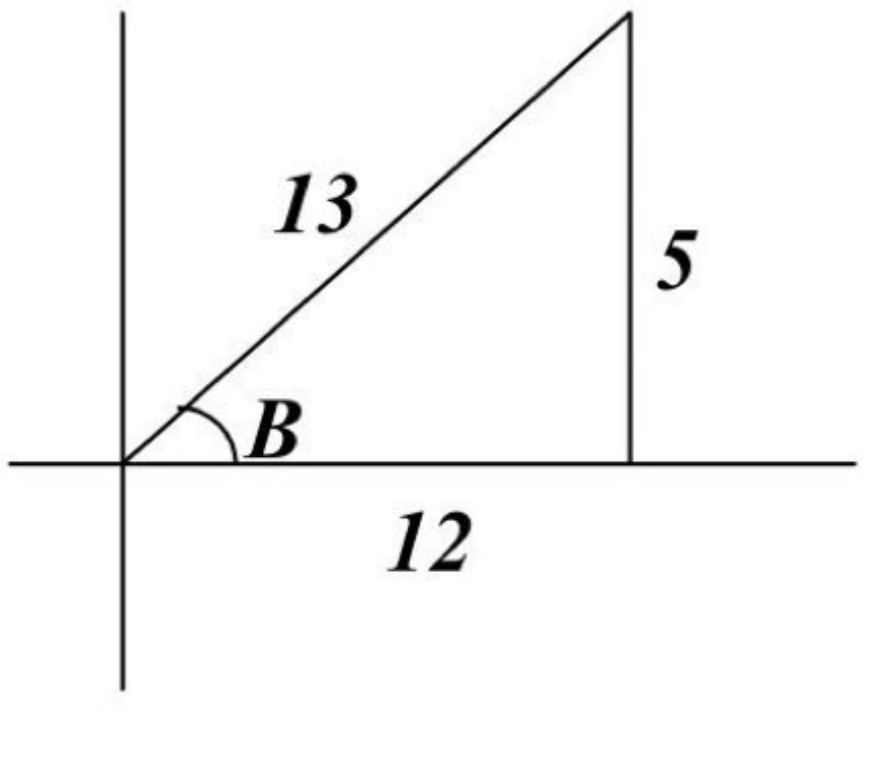
$= a^2(Sin^2\theta + Cos^2\theta) + b^2(Cos^2\theta + Sin^2\theta) = a^2 + b^2$

<u>EX-9</u>- If $Sin\,A = 4/5$ and $Cos\,B = 12/13$ ,where $A$ is obtuse and $B$ is acute . Find , without tables , the values of :

a) $Sin(A - B)$ , b) $\tan(A - B)$ , c) $\tan(A + B)$ .

<u>Sol.</u> -



$$a)\quad Sin(A-B) = SinA.CosB - CosA.SinB$$
$$= \frac{4}{5}.\frac{12}{13} + \frac{3}{5}.\frac{5}{13} = \frac{63}{65}$$

$$b)\quad tan(A-B) = \frac{tan\,A - tan\,B}{1 + tan\,A.tan\,B}$$
$$= \frac{-\frac{4}{3} - \frac{5}{12}}{1 - \frac{4}{3}.\frac{5}{12}} = -\frac{63}{16}$$

$$c)\quad tan(A+B) = \frac{tan\,A + tan\,B}{1 - tan\,A.tan\,B}$$
$$= \frac{-\frac{4}{3} + \frac{5}{12}}{1 + \frac{4}{3}.\frac{5}{12}} = \frac{33}{56}$$

***<u>EX-10</u>*** **– Prove the following identities:**

$$a)\quad Sin(A+B) + Sin(A-B) = 2.SinA.CosB$$

$$b)\quad tan\,A + tan\,B = \frac{Sin(A+B)}{CosA.CosB}$$

$$c)\quad Sec(A+B) = \frac{SecA.SecB.CscA.CscB}{CscA.CscB - SecA.SecB}$$

$$d)\quad \frac{Sin\,2\theta + Cos\,2\theta + 1}{Sin\,2\theta - Cos\,2\theta + 1} = Cot\,\theta$$

**Sol.-**

a) L.H.S. $= Sin(A+B) + Sin(A-B)$

$= SinA.CosB + CosA.SinB + SinA.CosB - CosA.SinB$

$= 2.SinA.CosB = R.H.S.$

b) R.H.S. $= \dfrac{Sin(A+B)}{CosA.CosB} = \dfrac{SinA.CosB + CosA.SinB}{CosA.CosB}$

$= \tan A + \tan B = L.H.S.$

c) R.H.S $= \dfrac{SecA.SecB.CscA.CscB}{CscA.CscB - SecA.SecB} = \dfrac{\frac{1}{CosA}.\frac{1}{CosB}.\frac{1}{SinA}.\frac{1}{SinB}}{\frac{1}{SinA}.\frac{1}{SinB} - \frac{1}{CosA}.\frac{1}{CosB}}$

$= \dfrac{1}{CosA.CosB - SinA.SinB} = \dfrac{1}{Cos(A+B)}$

$= Sec(A+B) = L.H.S.$

d) L.H.S. $= \dfrac{Sin2\theta + Cos2\theta + 1}{Sin2\theta - Cos2\theta + 1} = \dfrac{2Sin\theta.Cos\theta + (Cos^2\theta - Sin^2\theta) + 1}{2Sin\theta.Cos\theta - (Cos^2\theta - Sin^2\theta) + 1}$

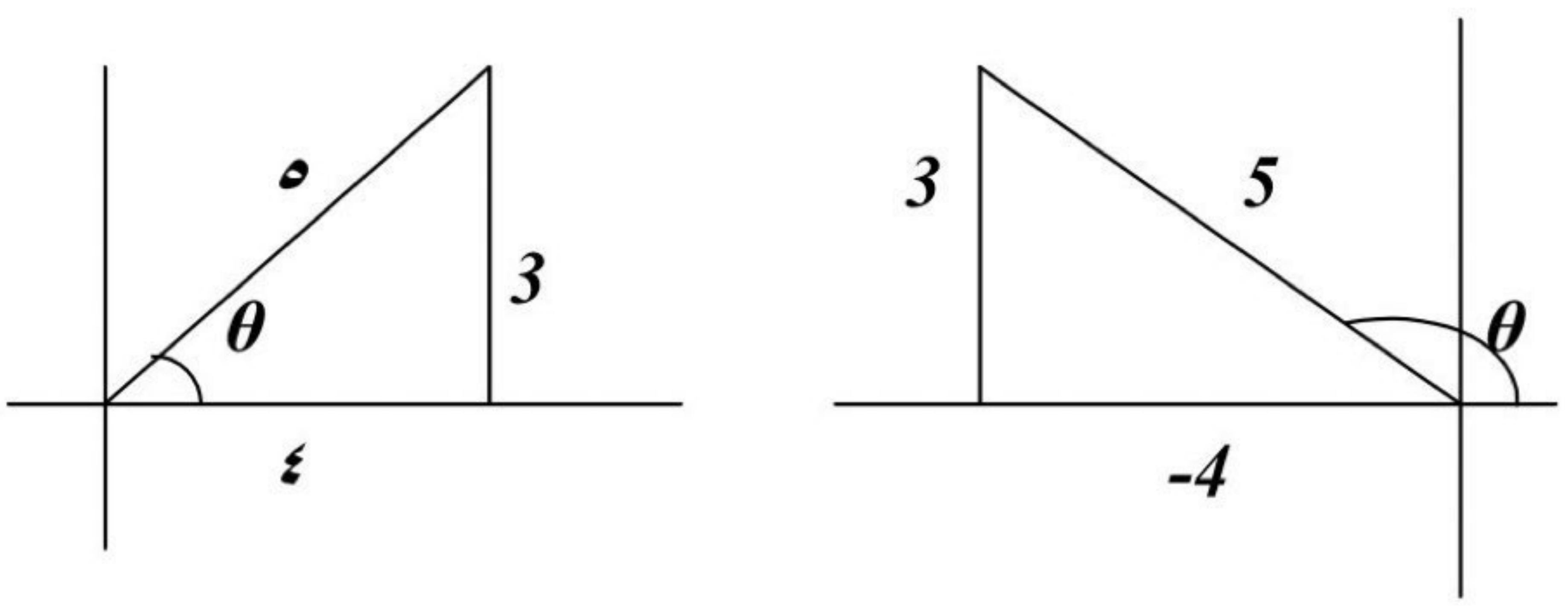
$= \dfrac{2Sin\theta.Cos\theta + 2Cos^2\theta}{2Sin\theta.Cos\theta + 2Sin^2\theta} = \dfrac{Cos\theta}{Sin\theta} = Cot\theta = R.H.S.$

**EX-11** – **Find , without using tables , the values of** $Sin\,2\theta$ **and** $Cos\,2\theta$**, when:**

**a)** $Sin\theta = 3/5$ , **b)** $Cos\,\theta = 12/13$ , **c)** $Sin\,\theta = -\sqrt{3}/2$ .

**Sol.** –

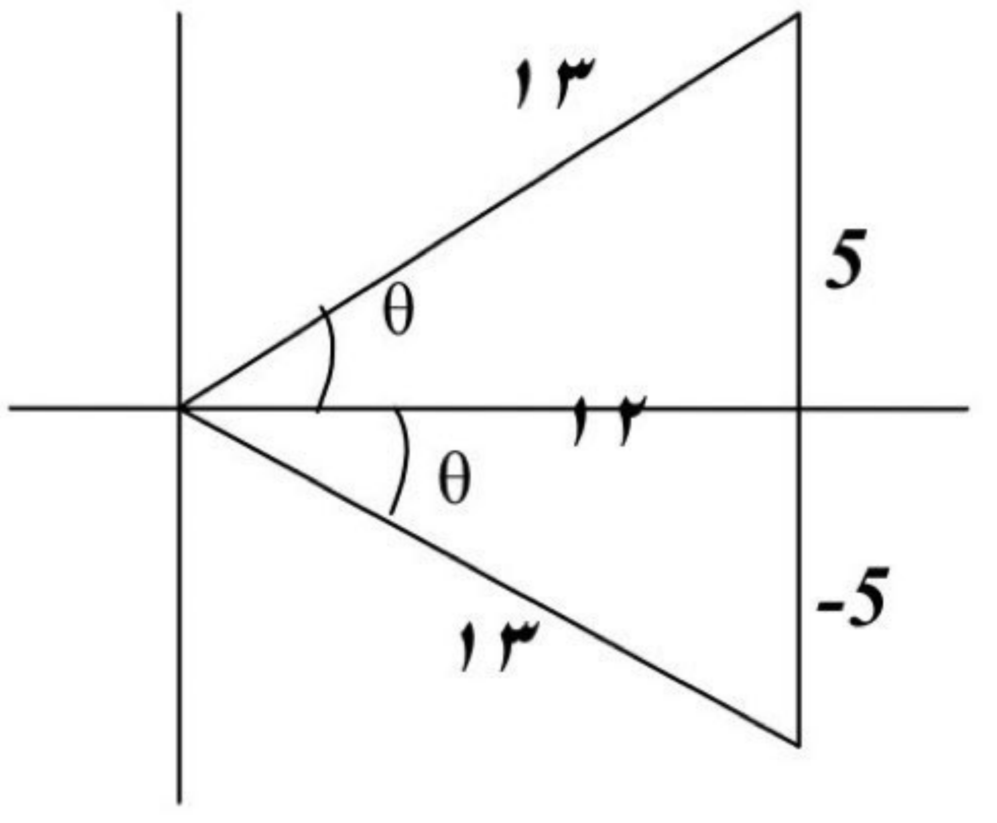
**a)**




$$Sin2\theta = 2.Sin\theta.Cos\theta = 2.\frac{3}{5}.(\mp\frac{4}{5}) = \mp\frac{24}{25}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{4}{5})^2 - (\frac{3}{5})^2 = \frac{7}{25}$$

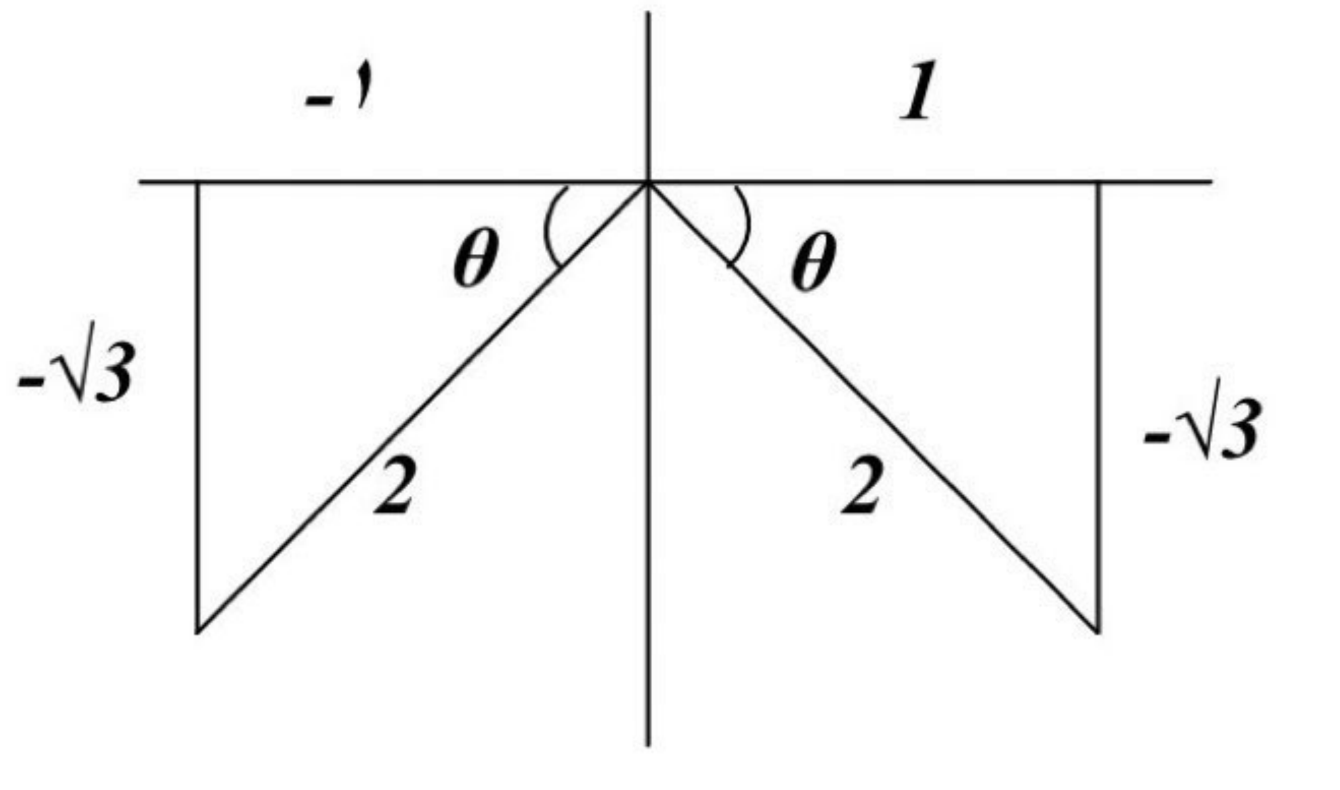
**b)**




$$Sin2\theta = 2.Sin\theta.Cos\theta = 2(\mp\frac{5}{13}).(\frac{12}{13}) = \mp\frac{120}{169}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\frac{12}{13})^2 - (\mp\frac{5}{13})^2 = \frac{119}{169}$$

**c)**




$$Sin2\theta = 2Sin\theta.Cos\theta = 2(-\frac{\sqrt{3}}{2}).(\mp\frac{1}{2}) = \pm\frac{\sqrt{3}}{2}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{1}{2})^2 - (-\frac{\sqrt{3}}{2})^2 = -\frac{1}{2}$$

***EX-12-*** **Solve the following equations for values of** $\theta$ **from** $0^o$ **to** $360^o$ **inclusive:**

**a)** $Cos\ 2\theta + Cos\ \theta + 1 = 0$ , **b)** $4\ tan\ \theta\ .\ tan\ 2\theta = 1$

***Sol.-***

$$a)\quad Cos2\theta + Cos\theta + 1 = 0 \Rightarrow 2Cos^2\theta - 1 + Cos\theta + 1 = 0$$

$$\Rightarrow Cos(2.Cos\theta + 1) = 0$$

$$either \quad Cos\theta = 0 \Rightarrow \theta = 90^o, 270^o$$

$$or \quad Cos\theta = -\frac{1}{2} \Rightarrow \theta = 120^o, 240^o$$

$$\theta = \{90^o, 120^o, 240^o, 270^o\}$$

$$b)\quad 4.\tan\theta.\tan 2\theta = 1 \Rightarrow 4.\tan\theta.\frac{2\tan\theta}{1-\tan^2\theta} = 1$$

$$\Rightarrow 9\tan^2\theta = 1$$

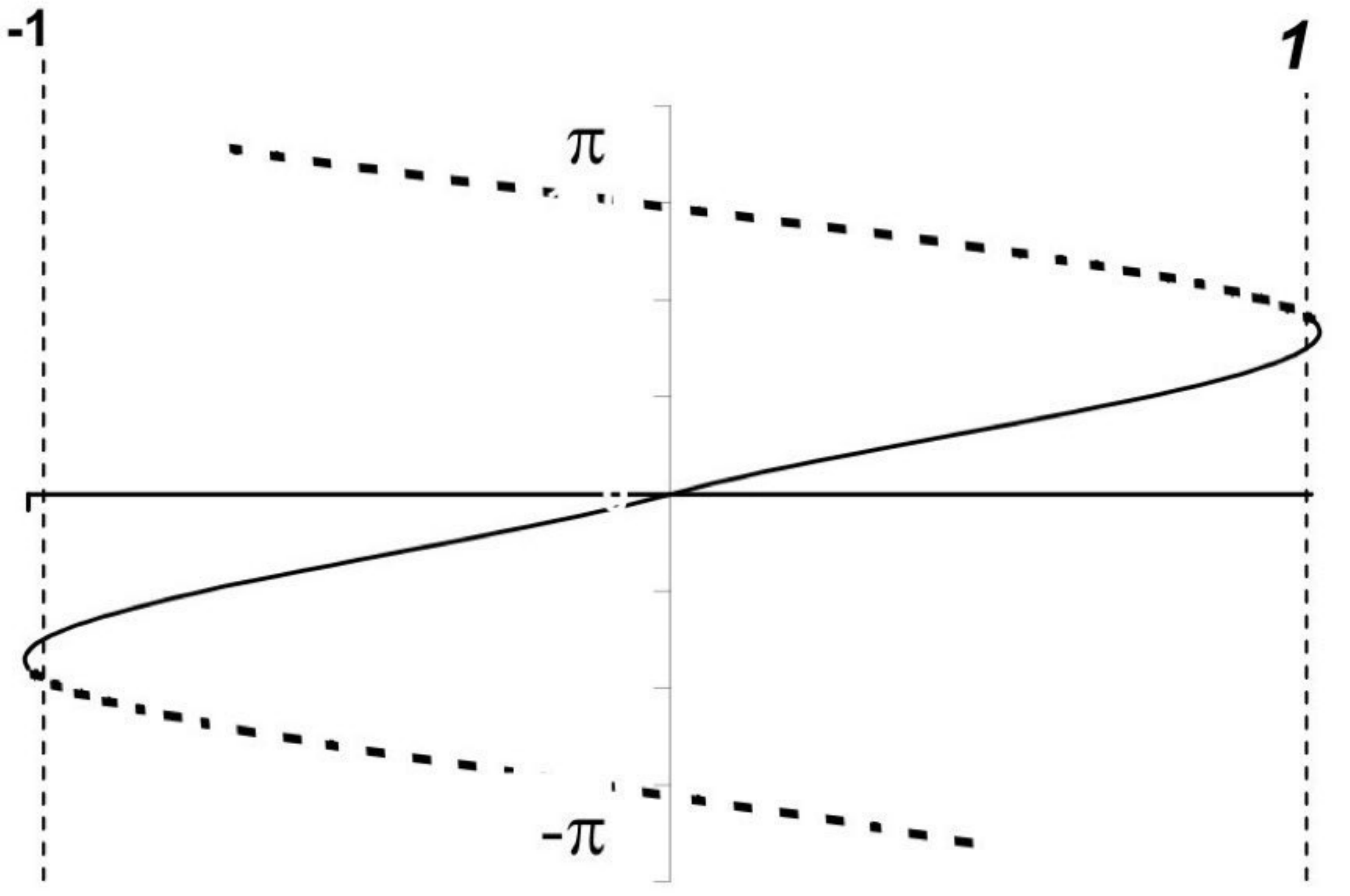
$$either \quad \tan\theta = \frac{1}{3} \Rightarrow \theta = 18.4^o, 198.4^o$$

$$or \quad \tan\theta = -\frac{1}{3} \Rightarrow \theta = 161.6^o, 341.6^o$$

$$\theta = \{18.4^o, 161.6^o, 198.4^o, 341.6^o\}$$

**2-3- <u>The inverse trigonometric functions</u> : The inverse trigonometric functions arise in problems that require finding angles from side measurements in triangles :**

$$y = Sinx \quad \Leftrightarrow \quad x = Sin^{-1}y$$




$$y = Sin^{-1}x \quad D_x : -1 \le x \le 1$$

$$R_y : -90 \le y \le 90$$