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LECTURE: (3)

Subject: Derivatives

Level: First

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Chapter three Derivatives

Let $y = f(x)$ be a function of x . If the limit :

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

exists and is finite, we call this limit the derivative of f at x and say that f is differentiable at x .

EX-1 – Find the derivative of the function : $f(x) = \frac{1}{\sqrt{2x+3}}$

Sol.:

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{\sqrt{2(x + \Delta x) + 3}} - \frac{1}{\sqrt{2x + 3}}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{2x + 3} - \sqrt{2(x + \Delta x) + 3}}{\Delta x \cdot \sqrt{2(x + \Delta x) + 3} \sqrt{2x + 3}} \cdot \frac{\sqrt{2x + 3} + \sqrt{2(x + \Delta x) + 3}}{\sqrt{2x + 3} + \sqrt{2(x + \Delta x) + 3}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(2x + 3) - (2(x + \Delta x) + 3)}{\Delta x \cdot \sqrt{2(x + \Delta x) + 3} \sqrt{2x + 3} (\sqrt{2x + 3} + \sqrt{2(x + \Delta x) + 3})} \\ &= \frac{-2}{(2x + 3)(\sqrt{2x + 3} + \sqrt{2x + 3})} = -\frac{1}{\sqrt{(2x + 3)^3}} \end{aligned}$$

Rules of derivatives : Let c and n are constants, u , v and w are differentiable functions of x :

1. $\frac{d}{dx} c = 0$

2. $\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx} \Rightarrow \frac{d}{dx} \left(\frac{1}{u} \right) = -\frac{1}{u^2} \frac{du}{dx}$

3. $\frac{d}{dx} cu = c \frac{du}{dx}$

4. $\frac{d}{dx} (u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx}$; $\frac{d}{dx} (u \mp v \mp w) = \frac{du}{dx} \mp \frac{dv}{dx} \mp \frac{dw}{dx}$

5. $\frac{d}{dx} (u.v) = u \cdot \frac{dv}{dx} + v \frac{du}{dx}$

$$\text{and } \frac{d}{dx}(u.v.w) = u.v \frac{dw}{dx} + u.w \frac{dv}{dx} + v.w \frac{du}{dx}$$

$$6. \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \quad \text{where } v \neq 0$$

EX-2- Find $\frac{dy}{dx}$ for the following functions :

$$a) y = (x^2 + 1)^5$$

$$b) y = [(5 - x)(4 - 2x)]^2$$

$$c) y = (2x^3 - 3x^2 + 6x)^{-5}$$

$$d) y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$$

$$e) y = \frac{(x^2 + x)(x^2 - x + 1)}{x^3}$$

$$f) y = \frac{x^2 - 1}{x^2 + x - 2}$$

Sol.-

$$a) \frac{dy}{dx} = 5(x^2 + 1)^4 \cdot 2x = 10x(x^2 + 1)^4$$

$$b) \frac{dy}{dx} = 2[(5 - x)(4 - 2x)][-2(5 - x) - (4 - 2x)]$$

$$= 8(5 - x)(2 - x)(2x - 7)$$

$$c) \frac{dy}{dx} = -5(2x^3 - 3x^2 + 6x)^{-6} (6x^2 - 6x + 6)$$

$$= -30(2x^3 - 3x^2 + 6x)^{-6} (x^2 - x + 1)$$

$$d) y = 12x^{-1} - 4x^{-3} + 3x^{-4} \Rightarrow \frac{dy}{dx} = -12x^{-2} + 12x^{-4} - 12x^{-5}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{12}{x^2} + \frac{12}{x^4} - \frac{12}{x^5}$$

$$e) y = \frac{(x+1)(x^2 - x + 1)}{x^3} \Rightarrow$$

$$\frac{dy}{dx} = \frac{x^3[(x^2 - x + 1) + (x+1)(2x-1)] - 3x^2(x+1)(x^2 - x + 1)}{x^6} = -\frac{3}{x^4}$$

$$f) \frac{dy}{dx} = \frac{2x(x^2 + x - 2) - (x^2 - 1)(2x + 1)}{(x^2 + x - 2)^2} = \frac{x^2 - 2x + 1}{(x^2 + x - 2)^2}$$

The Chain Rule:

1. Suppose that $h = g \circ f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable function of x whose derivative at each value of x is :

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dx}{dt}$$

2. If y is a differentiable function of t and t is differentiable function of x , then y is a differentiable function of x :

$$y = g(t) \text{ and } t = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

EX-3 – Use the chain rule to express dy/dx in terms of x and y :

a) $y = \frac{t^2}{t^2 + 1}$ and $t = \sqrt{2x + 1}$
b) $y = \frac{1}{t^2 + 1}$ and $x = \sqrt{4t + 1}$
c) $y = \left(\frac{t-1}{t+1}\right)^2$ and $x = \frac{1}{t^2} - 1$ at $t = 2$
d) $y = 1 - \frac{1}{t}$ and $t = \frac{1}{1-x}$ at $x = 2$

Sol.-

a) $y = \frac{t^2}{t^2 + 1} \Rightarrow \frac{dy}{dt} = \frac{2t(t^2 + 1) - 2t \cdot t^2}{(t^2 + 1)^2} = \frac{2t}{(t^2 + 1)^2}$
 $t = (2x + 1)^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2} \cdot (2x + 1)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{2x + 1}}$
 $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2t}{(t^2 + 1)^2} \cdot \frac{1}{\sqrt{2x + 1}} = \frac{2\sqrt{2x + 1}}{((2x + 1) + 1)^2} \cdot \frac{1}{\sqrt{2x + 1}} = \frac{1}{2(x + 1)^2}$

$$\begin{aligned}
 b) \quad y &= (t^2 + 1)^{-1} \Rightarrow \frac{dy}{dx} = -2t(t^2 + 1)^{-2} = -\frac{2t}{(t^2 + 1)^2} \\
 x &= (4t + 1)^{\frac{1}{2}} \Rightarrow \frac{dx}{dt} = \frac{1}{2}(4t + 1)^{-\frac{1}{2}} \cdot 4 = \frac{2}{\sqrt{4t + 1}} \\
 \frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{2t}{(t^2 + 1)^2} \div \frac{2}{\sqrt{4t + 1}} = -\frac{t\sqrt{4t + 1}}{(t^2 + 1)^2} \\
 &= -\frac{x^2 - 1}{4} \cdot x \div \frac{1}{y^2} = -\frac{xy^2(x^2 - 1)}{4}
 \end{aligned}$$

$$\text{where } x = \sqrt{4t + 1} \Rightarrow t = \frac{x^2 - 1}{4}$$

$$\text{where } y = \frac{1}{t^2 + 1} \Rightarrow t^2 + 1 = \frac{1}{y}$$

$$\begin{aligned}
 c) \quad y &= \left(\frac{t-1}{t+1}\right)^2 \Rightarrow \frac{dy}{dt} = 2\left(\frac{t-1}{t+1}\right) \frac{t+1 - (t-1)}{(t+1)^2} = \frac{4(t-1)}{(t+1)^3} \\
 &\Rightarrow \left[\frac{dy}{dt}\right]_{t=2} = \frac{4(2-1)}{(2+1)^3} = \frac{4}{27}
 \end{aligned}$$

$$x = \frac{1}{t^2} - 1 \Rightarrow \frac{dx}{dt} = -\frac{2}{t^3} \Rightarrow \left[\frac{dx}{dt}\right]_{t=2} = -\frac{2}{2^3} = -\frac{1}{4}$$

$$\left[\frac{dy}{dx}\right]_{t=2} = \left[\frac{dy}{dt} \div \frac{dx}{dt}\right]_{t=2} = \frac{4}{27} \div \left(-\frac{1}{4}\right) = -\frac{16}{27}$$

$$d) \quad t = \frac{1}{1-x} = \frac{1}{1-2} = -1 \quad \text{at } x = 2$$

$$y = 1 - \frac{1}{t} \Rightarrow \frac{dy}{dt} = \frac{1}{t^2} \Rightarrow \left[\frac{dy}{dt}\right]_{t=-1} = \frac{1}{(-1)^2} = 1$$

$$t = (1-x)^{-1} \Rightarrow \frac{dt}{dx} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\Rightarrow \left[\frac{dt}{dx}\right]_{x=2} = \frac{1}{(1-2)^2} = 1$$

$$\left[\frac{dy}{dx}\right]_{x=2} = \left[\frac{dy}{dt}\right]_{x=2} \cdot \left[\frac{dt}{dx}\right]_{x=2} = 1 * 1 = 1$$

Higher derivatives : If a function $y = f(x)$ possesses a derivative at every point of some interval, we may form the function $f'(x)$ and talk

about its derivative, if it has one. The procedure is formally identical with that used before, that is :

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}$$

if the limit exists.

This derivative is called the second derivative of y with respect to x . It is written in a number of ways, for example,

$$y'', f''(x), \text{ or } \frac{d^2 f(x)}{dx^2}.$$

In the same manner we may define third and higher derivatives, using similar notations. The n th derivative may be written :

$$y^{(n)}, f^{(n)}(x), \frac{d^n y}{dx^n}.$$

EX-4- Find all derivatives of the following function :

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol.-

$$\begin{aligned} \frac{dy}{dx} &= 9x^2 - 8x + 7, & \frac{d^2 y}{dx^2} &= 18x - 8 \\ \frac{d^3 y}{dx^3} &= 18, & \frac{d^4 y}{dx^4} &= 0 = \frac{d^5 y}{dx^5} = \dots \end{aligned}$$

Ex-5 – Find the third derivative of the following function :

$$y = \frac{1}{x} + \sqrt{x^3}$$

Sol.-

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{x^2} + \frac{3}{2}x^{\frac{1}{2}} \\ \frac{d^2 y}{dx^2} &= \frac{2}{x^3} + \frac{3}{4}x^{-\frac{1}{2}} \\ \frac{d^3 y}{dx^3} &= -\frac{6}{x^4} - \frac{3}{8}x^{-\frac{3}{2}} \quad \Rightarrow \quad \frac{d^3 y}{dx^3} = -\frac{6}{x^4} - \frac{3}{8\sqrt{x^3}} \end{aligned}$$

Implicit Differentiation: If the formula for f is an algebraic combination of powers of x and y . To calculate the derivatives of these implicitly defined functions, we simply differentiate both sides of the defining equation with respect to x .

EX-6- Find $\frac{dy}{dx}$ for the following functions:

$$a) x^2 \cdot y^2 = x^2 + y^2$$

$$b) (x + y)^3 + (x - y)^3 = x^4 + y^4$$

$$c) \frac{x - y}{x - 2y} = 2 \text{ at } P(3,1)$$

$$d) xy + 2x - 5y = 2 \text{ at } P(3,2)$$

Sol.

$$a) x^2 \left(2y \frac{dy}{dx} \right) + y^2 (2x) = 2x + 2y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2 y - y}$$

$$b) 3(x + y)^2 \left(1 + \frac{dy}{dx} \right) + 3(x - y)^2 \left(1 - \frac{dy}{dx} \right) = 4x^3 + 4y^3 \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x + y)^2 - 3(x - y)^2}{3(x + y)^2 - 3(x - y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

$$c) \frac{(x - 2y) \left(1 - \frac{dy}{dx} \right) - (x - y) \left(1 - 2 \frac{dy}{dx} \right)}{(x - 2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[\frac{dy}{dx} \right]_{(3,1)} = \frac{1}{3}$$

$$d) x \frac{dy}{dx} + y + 2 - 5 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y + 2}{5 - x} \Rightarrow \left[\frac{dy}{dx} \right]_{(3,2)} = \frac{2 + 2}{5 - 3} = 2$$

Exponential functions: If u is any differentiable function of x , then :

$$7) \frac{d}{dx} a^u = a^u \cdot \ln a \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$$

EX-7 – Find $\frac{dy}{dx}$ for the following functions :

a) $y = 2^{3x}$

b) $y = 2^x \cdot 3^x$

c) $y = (2^x)^2$

d) $y = x \cdot 2^{x^2}$

e) $y = e^{(x+e^{5x})}$

f) $y = e^{\sqrt{1+5x^2}}$

Sol. –

a) $y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x} \cdot 3 \ln 2$

b) $y = 2^x \cdot 3^x \Rightarrow y = 6^x \Rightarrow \frac{dy}{dx} = 6^x \cdot \ln 6$

c) $y = (2^x)^2 \Rightarrow y = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x} \ln 2 \cdot 2 = 2^{2x+1} \ln 2$

d) $y = x \cdot 2^{x^2} \Rightarrow \frac{dy}{dx} = x \cdot 2^{x^2} \ln 2 \cdot 2x + 2^{x^2} = 2^{x^2} (2x^2 \ln 2 + 1)$

e) $y = e^{(x+e^{5x})} \Rightarrow \frac{dy}{dx} = e^{(x+e^{5x})} (1 + 5e^{5x})$

f) $y = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}} \cdot \frac{1}{2} (1+5x^2)^{-\frac{1}{2}} \cdot 10x = e^{\sqrt{1+5x^2}} \frac{5x}{\sqrt{1+5x^2}}$

Logarithm functions : If u is any differentiable function of x , then :

8) $\frac{d}{dx} \log_a u = \frac{1}{u \cdot \ln a} \cdot \frac{du}{dx}$ and $\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$

EX-8 – Find $\frac{dy}{dx}$ for the following functions :

a) $y = \log_{10} e^x$

b) $y = \log_5 (x+1)^2$

c) $y = \log_2 (3x^2 + 1)^3$

d) $y = [\ln(x^2 + 2)^2]^3$

e) $y + \ln(xy) = 1$

f) $y = \frac{(2x^3 - 4)^{\frac{2}{3}} \cdot (2x^2 + 3)^{\frac{5}{2}}}{(7x^3 + 4x - 3)^2}$

Sol. –

$$\begin{aligned}
a) \quad y &= \log_{10} e^x \Rightarrow y = x \log_{10} e \Rightarrow \frac{dy}{dx} = \log_{10} e = \frac{\ln e}{\ln 10} = \frac{1}{\ln 10} \\
b) \quad y &= \log_5 (x+1)^2 = 2 \log_5 (x+1) \Rightarrow \frac{dy}{dx} = \frac{2}{(x+1) \ln 5} \\
c) \quad y &= 3 \log_2 (3x^2 + 1) \Rightarrow \frac{dy}{dx} = \frac{3}{3x^2 + 1} \cdot \frac{6x}{\ln 2} = \frac{18x}{(3x^2 + 1) \ln 2} \\
d) \quad \frac{dy}{dx} &= 3 [2 \ln(x^2 + 2)]^2 \cdot \frac{2}{x^2 + 2} \cdot 2x = \frac{48x [\ln(x^2 + 2)]^2}{x^2 + 2} \\
e) \quad y + \ln x + \ln y &= 1 \Rightarrow \frac{dy}{dx} + \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x(y+1)} \\
f) \quad \ln y &= \frac{2}{3} \ln(2x^3 - 4) + \frac{5}{2} \ln(2x^2 + 3) - 2 \ln(7x^3 + 4x - 3) \\
&\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{3} \cdot \frac{6x^2}{2x^3 - 4} + \frac{5}{2} \cdot \frac{4x}{2x^2 + 3} - 2 \cdot \frac{21x^2 + 4}{7x^3 + 4x - 3} \\
&\Rightarrow \frac{dy}{dx} = 2y \left[\frac{2x^2}{2x^3 - 4} + \frac{5x}{2x^2 + 3} - \frac{21x^2 + 4}{7x^3 + 4x - 3} \right]
\end{aligned}$$

Trigonometric functions : If u is any differentiable function of x , then :

$$\begin{aligned}
9) \quad \frac{d}{dx} \sin u &= \cos u \cdot \frac{du}{dx} \\
10) \quad \frac{d}{dx} \cos u &= -\sin u \cdot \frac{du}{dx} \\
11) \quad \frac{d}{dx} \tan u &= \sec^2 u \cdot \frac{du}{dx} \\
12) \quad \frac{d}{dx} \cot u &= -\csc^2 u \cdot \frac{du}{dx} \\
13) \quad \frac{d}{dx} \sec u &= \sec u \cdot \tan u \cdot \frac{du}{dx} \\
14) \quad \frac{d}{dx} \csc u &= -\csc u \cdot \cot u \cdot \frac{du}{dx}
\end{aligned}$$

EX-9- Find $\frac{dy}{dx}$ for the following functions :

$$a) y = \tan(3x^2)$$

$$b) y = (\csc x + \cot x)^2$$

$$c) y = 2\sin \frac{x}{2} - x \cos \frac{x}{2}$$

$$d) y = \tan^2(\cos x)$$

$$e) x + \tan(xy) = 0$$

$$f) y = \sec^4 x - \tan^4 x$$

Sol.-

$$a) \frac{dy}{dx} = \sec^2(3x^2) \cdot 6x = 6x \cdot \sec^2(3x^2)$$

$$b) \frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x \cdot \cot x - \csc^2 x) = -2 \csc x \cdot (\csc x + \cot x)^2$$

$$c) \frac{dy}{dx} = 2 \cos \frac{x}{2} \cdot \frac{1}{2} - \left[x \left(-\sin \frac{x}{2} \right) \cdot \frac{1}{2} + \cos \frac{x}{2} \right] = \frac{x}{2} \cdot \sin \frac{x}{2}$$

$$d) \frac{dy}{dx} = 2 \tan(\cos x) \cdot \sec^2(\cos x) \cdot (-\sin x) = -2 \sin x \cdot \tan(\cos x) \cdot \sec^2(\cos x)$$

$$e) 1 + \sec^2(xy) \cdot \left(x \frac{dy}{dx} + y \right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1 + y \cdot \sec^2(xy)}{x \cdot \sec^2(xy)} = -\frac{\cos^2(xy) + y}{x}$$

$$f) \frac{dy}{dx} = 4 \sec^3 x \cdot \sec x \cdot \tan x - 4 \tan^3 x \cdot \sec^2 x = 4 \tan x \cdot \sec^2 x$$

EX-10- Prove that :

$$a) \frac{d}{dx} \tan u = \sec^2 u \cdot \frac{du}{dx}$$

$$b) \frac{d}{dx} \sec u = \sec u \cdot \tan u \cdot \frac{du}{dx}$$

Proof:

$$\begin{aligned} a) \text{ L.H.S.} &= \frac{d}{dx} \tan u = \frac{d}{dx} \frac{\sin u}{\cos u} = \frac{\cos u \cdot \cos u \cdot \frac{du}{dx} - \sin u \cdot (-\sin u) \frac{du}{dx}}{\cos^2 u} \\ &= \frac{\cos^2 u + \sin^2 u}{\cos^2 u} \cdot \frac{du}{dx} = \frac{1}{\cos^2 u} \cdot \frac{du}{dx} = \sec^2 u \cdot \frac{du}{dx} = \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} b) \text{ L.H.S.} &= \frac{d}{dx} \sec u = \frac{d}{dx} \frac{1}{\cos u} = -\frac{1}{\cos^2 u} (-\sin u) \frac{du}{dx} \\ &= \frac{1}{\cos u} \cdot \frac{\sin u}{\cos u} \cdot \frac{du}{dx} = \sec u \cdot \tan u \cdot \frac{du}{dx} = \text{R.H.S.} \end{aligned}$$

The inverse trigonometric functions : If u is any differentiable function

of x , then :

$$15) \quad \frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \quad -1 < u < 1$$

$$16) \quad \frac{d}{dx} \cos^{-1} u = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \quad -1 < u < 1$$

$$17) \quad \frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$$

$$18) \quad \frac{d}{dx} \cot^{-1} u = -\frac{1}{1+u^2} \frac{du}{dx}$$

$$19) \quad \frac{d}{dx} \sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx} \quad |u| > 1$$

$$20) \quad \frac{d}{dx} \csc^{-1} u = -\frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx} \quad |u| > 1$$

EX-11- Find $\frac{dy}{dx}$ in each of the following functions :

$$a) \quad y = \cot^{-1} \frac{2}{x} + \tan^{-1} \frac{x}{2}$$

$$b) \quad y = \sin^{-1} \frac{x-1}{x+1}$$

$$c) \quad y = x \cdot \cos^{-1} 2x - \frac{1}{2} \sqrt{1-4x^2}$$

$$d) \quad y = \sec^{-1} 5x$$

$$e) \quad y = x \cdot \ln(\sec^{-1} x)$$

$$f) \quad y = 3^{\sin^{-1} 2x}$$

Sol. -

$$a) \quad \frac{dy}{dx} = -\frac{1}{1+\left(\frac{2}{x}\right)^2} 2 \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2} = \frac{4}{4+x^2}$$

$$b) \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{(x+1) \cdot 1 - (x-1) \cdot 1}{(x+1)^2} = \frac{1}{(x+1)\sqrt{x}}$$

$$c) \quad \frac{dy}{dx} = x \cdot \frac{-2}{\sqrt{1-4x^2}} + \cos^{-1} 2x - \frac{1}{4} \cdot \frac{-8x}{\sqrt{1-4x^2}} = \cos^{-1} 2x$$

$$d) \quad \frac{dy}{dx} = \frac{5}{|5x|\sqrt{25x^2-1}} = \frac{1}{|x|\sqrt{25x^2-1}}$$

$$e) \frac{dy}{dx} = \frac{x}{\sec^{-1} x} \cdot \frac{1}{|x|\sqrt{x^2-1}} + \ln(\sec^{-1} x) = \frac{1}{\sqrt{x^2-1} \cdot \sec^{-1} x} + \ln(\sec^{-1} x)$$

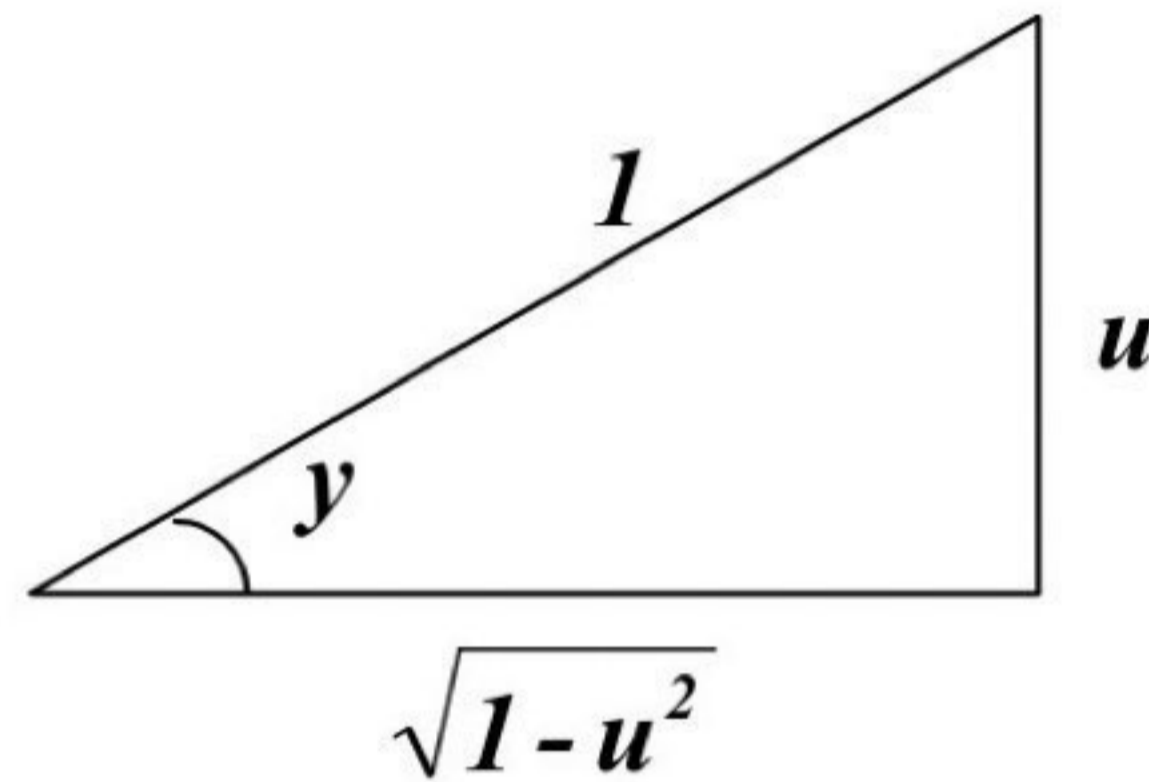
$$f) \frac{dy}{dx} = 3^{\sin^{-1} 2x} \cdot \ln 3 \cdot \frac{2}{\sqrt{1-4x^2}}$$

EX-12- Prove that :

$$a) \frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

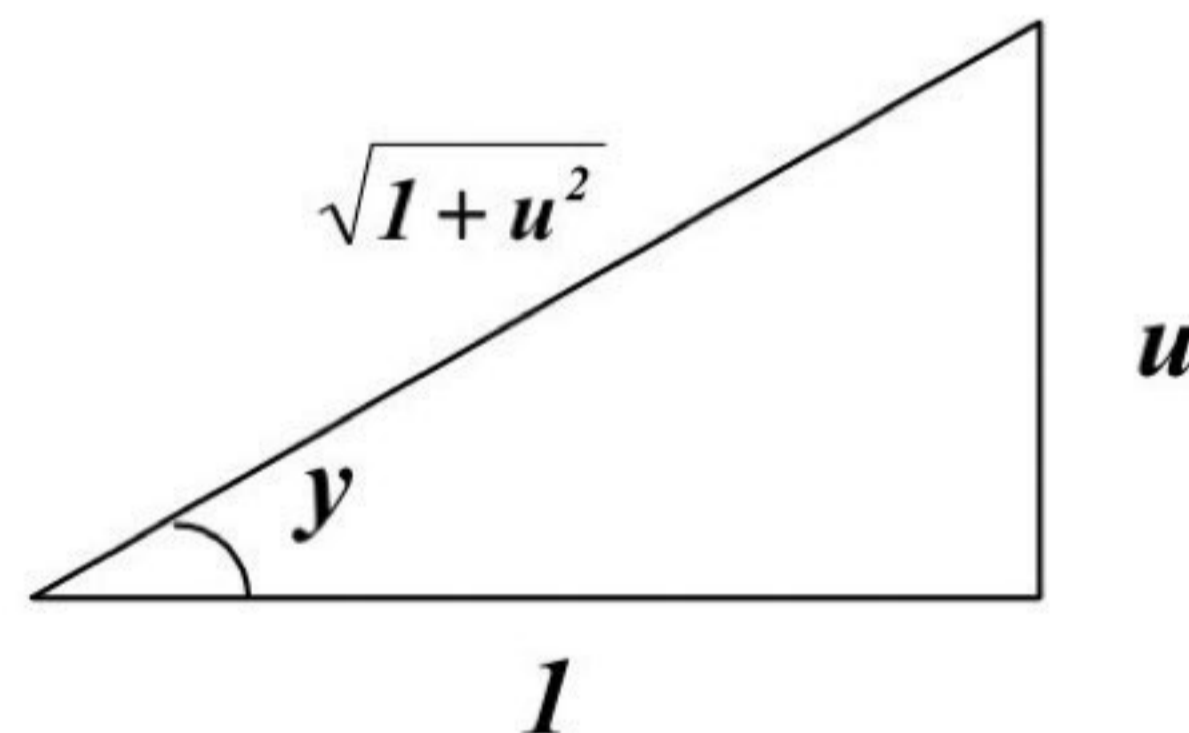
$$b) \frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$$

Proof: a)



$$\begin{aligned} \text{Let } y = \sin^{-1} u \Rightarrow u = \sin y \Rightarrow \frac{du}{dx} &= \cos y \cdot \frac{dy}{dx} = \sqrt{1-u^2} \frac{dy}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \Rightarrow \frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \end{aligned}$$

b)



$$\begin{aligned} \text{Let } y = \tan^{-1} u \Rightarrow u = \tan y \Rightarrow \frac{du}{dx} &= \sec^2 y \cdot \frac{dy}{dx} = (\sqrt{1+u^2})^2 \frac{dy}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{1+u^2} \cdot \frac{du}{dx} \Rightarrow \frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \cdot \frac{du}{dx} \end{aligned}$$

Hyperbolic functions : If u is any differentiable function of x , then :

$$21) \quad \frac{d}{dx} \sinh u = \cosh u \cdot \frac{du}{dx}$$

$$22) \quad \frac{d}{dx} \cosh u = \sinh u \cdot \frac{du}{dx}$$

$$23) \quad \frac{d}{dx} \tanh u = \operatorname{sech}^2 u \cdot \frac{du}{dx}$$

$$24) \quad \frac{d}{dx} \coth u = -\operatorname{csch}^2 u \cdot \frac{du}{dx}$$

$$25) \quad \frac{d}{dx} \operatorname{sech} u = -\operatorname{sech} u \cdot \tanh u \cdot \frac{du}{dx}$$

$$26) \quad \frac{d}{dx} \operatorname{csch} u = -\operatorname{csch} u \cdot \coth u \cdot \frac{du}{dx}$$

EX-13 - Find $\frac{dy}{dx}$ for the following functions :

$$a) \quad y = \coth(\tan x)$$

$$b) \quad y = \sin^{-1}(\tanh x)$$

$$c) \quad y = \ln \left| \tanh \frac{x}{2} \right|$$

$$d) \quad y = x \cdot \sinh 2x - \frac{1}{2} \cdot \cosh 2x$$

$$e) \quad y = \operatorname{sech}^3 x$$

$$f) \quad y = \operatorname{csch}^2 x$$

Sol. -

$$a) \quad \frac{dy}{dx} = -\operatorname{csch}^2(\tan x) \cdot \sec^2 x$$

$$b) \quad \frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1 - \tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}} = \operatorname{sech} x$$

$$c) \quad \frac{dy}{dx} = \frac{1}{\tanh \frac{x}{2}} \operatorname{sech}^2 \frac{x}{2} \cdot \frac{1}{2} = \frac{1}{2 \cdot \frac{\sinh \frac{x}{2}}{\cosh \frac{x}{2}}} \cdot \frac{\cosh^2 \frac{x}{2}}{2} = \frac{1}{2 \sinh \frac{x}{2} \cdot \cosh \frac{x}{2}} = \frac{1}{\sinh x} = \operatorname{csch} x$$

$$d) \frac{dy}{dx} = x \cosh 2x \cdot 2 + \sinh 2x - \frac{1}{2} \sinh 2x \cdot 2 = 2x \cosh 2x$$

$$e) \frac{dy}{dx} = 3 \operatorname{sech}^2 x (-\operatorname{sech} x \cdot \tanh x) = -3 \operatorname{sech}^3 x \cdot \tanh x$$

$$f) \frac{dy}{dx} = 2 \operatorname{csch} x (-\operatorname{csch} x \cdot \coth x) = -2 \operatorname{csch}^2 x \cdot \coth x$$

EX-14- Show that the functions :

$$x = -\frac{2}{\sqrt{3}} \sinh \frac{t}{\sqrt{3}} \quad \text{and} \quad y = \frac{1}{\sqrt{3}} \sinh \frac{t}{\sqrt{3}} + \cosh \frac{t}{\sqrt{3}}$$

Taken together, satisfy the differential equations :

$$i) \frac{dx}{dt} + 2 \frac{dy}{dt} + x = 0 \quad \text{and} \quad ii) \frac{dx}{dt} - \frac{dy}{dt} + y = 0$$

Proof-

$$x = -\frac{2}{\sqrt{3}} \sinh \frac{t}{\sqrt{3}} \Rightarrow \frac{dx}{dt} = -\frac{2}{3} \cosh \frac{t}{\sqrt{3}}$$

$$y = \frac{1}{\sqrt{3}} \sinh \frac{t}{\sqrt{3}} + \cosh \frac{t}{\sqrt{3}} \Rightarrow \frac{dy}{dt} = \frac{1}{3} \cosh \frac{t}{\sqrt{3}} + \frac{1}{\sqrt{3}} \sinh \frac{t}{\sqrt{3}}$$

$$i) \frac{dx}{dt} + 2 \frac{dy}{dt} + x = -\frac{2}{3} \cosh \frac{t}{\sqrt{3}} + \frac{2}{3} \cosh \frac{t}{\sqrt{3}} + \frac{2}{\sqrt{3}} \sinh \frac{t}{\sqrt{3}} - \frac{2}{\sqrt{3}} \sinh \frac{t}{\sqrt{3}} = 0$$

$$ii) \frac{dx}{dt} - \frac{dy}{dt} + y = -\frac{2}{3} \cosh \frac{t}{\sqrt{3}} - \frac{1}{3} \cosh \frac{t}{\sqrt{3}} - \frac{1}{\sqrt{3}} \sinh \frac{t}{\sqrt{3}} + \frac{1}{\sqrt{3}} \sinh \frac{t}{\sqrt{3}} + \cosh \frac{t}{\sqrt{3}} = 0$$

EX-15 - Prove that :

$$a) \frac{d}{dx} \tanh u = \operatorname{sech}^2 u \cdot \frac{du}{dx} \quad \text{and} \quad b) \frac{d}{dx} \operatorname{sech} u = -\operatorname{sech} u \cdot \tanh u \cdot \frac{du}{dx}$$

Proof-

$$a) \frac{d}{dx} \tanh u = \frac{d}{dx} \left(\frac{\sinh u}{\cosh u} \right) = \frac{\cosh u \cdot \cosh u \cdot \frac{du}{dx} - \sinh u \cdot \sinh u \cdot \frac{du}{dx}}{\cosh^2 u}$$

$$= \frac{(\cosh^2 u - \sinh^2 u) \frac{du}{dx}}{\cosh^2 u} = \frac{1}{\cosh^2 u} \cdot \frac{du}{dx} = \operatorname{sech}^2 u \cdot \frac{du}{dx}$$

$$b) \frac{d}{dx} \frac{1}{\cosh u} = -\frac{1}{\cosh^2 u} \cdot \sinh u \cdot \frac{du}{dx} = -\operatorname{sech} u \cdot \tanh u \cdot \frac{du}{dx}$$

The inverse hyperbolic functions : If u is any differentiable function of x , then :

$$27) \quad \frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$$

$$28) \quad \frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}$$

$$29) \quad \frac{d}{dx} \tanh^{-1} u = \frac{1}{1-u^2} \frac{du}{dx} \quad |u| < 1$$

$$30) \quad \frac{d}{dx} \coth^{-1} u = \frac{1}{1-u^2} \frac{du}{dx} \quad |u| > 1$$

$$31) \quad \frac{d}{dx} \operatorname{sech}^{-1} u = -\frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}$$

$$32) \quad \frac{d}{dx} \operatorname{csch}^{-1} u = -\frac{1}{|u|\sqrt{1+u^2}} \frac{du}{dx}$$

EX-16 - Find $\frac{dy}{dx}$ for the following functions :

$$a) \quad y = \cosh^{-1}(\sec x)$$

$$b) \quad y = \tanh^{-1}(\cos x)$$

$$c) \quad y = \coth^{-1}(\sec x)$$

$$d) \quad y = \operatorname{sech}^{-1}(\sin 2x)$$

Sol.-

$$a) \quad \frac{dy}{dx} = \frac{\sec x \cdot \tan x}{\sqrt{\sec^2 x - 1}} = \frac{\sec x \cdot \tan x}{\sqrt{\tan^2 x}} = \sec x \quad \text{where } \tan x > 0$$

$$b) \quad \frac{dy}{dx} = \frac{-\sin x}{1 - \cos^2 x} = \frac{-\sin x}{\sin^2 x} = -\operatorname{csc} x$$

$$c) \quad \frac{dy}{dx} = \frac{\sec x \cdot \tan x}{1 - \sec^2 x} = \frac{\sec x \cdot \tan x}{-\tan^2 x} = -\operatorname{csc} x$$

$$d) \quad \frac{dy}{dx} = -\frac{2 \cdot \cos 2x}{\sin 2x \cdot \sqrt{1 - \sin^2 2x}} = -2 \operatorname{csc} 2x \quad \text{where } \cos 2x > 0$$

EX-17 - Verify the following formulas :

$$a) \quad \frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$b) \quad \frac{d}{dx} \tanh^{-1} u = \frac{1}{1-u^2} \cdot \frac{du}{dx} \quad |u| < 1$$

Proof

a) Let $y = \cosh^{-1} u \Rightarrow u = \cosh y$

$$\frac{du}{dx} = \sinh y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y} \cdot \frac{du}{dx}$$

$$\cosh^2 y - \sinh^2 y = 1 \Rightarrow u^2 - \sinh^2 y = 1 \Rightarrow \sinh y = \sqrt{u^2 - 1}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{u^2 - 1}} \cdot \frac{du}{dx} \Rightarrow \frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

b) Let $y = \tanh^{-1} u \Rightarrow u = \tanh y$

$$\frac{du}{dx} = \operatorname{sech}^2 y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} \cdot \frac{du}{dx}$$

$$\operatorname{sech}^2 y + \tanh^2 y = 1 \Rightarrow \operatorname{sech}^2 y + u^2 = 1 \Rightarrow \operatorname{sech}^2 y = 1 - u^2$$

$$\frac{dy}{dx} = \frac{1}{1 - u^2} \cdot \frac{du}{dx} \Rightarrow \frac{d}{dx} \tanh^{-1} u = \frac{1}{1 - u^2} \cdot \frac{du}{dx}$$

The derivatives of functions like u^v : Where u and v are differentiable functions of x , are found by logarithmic differentiation :

$$\text{Let } y = u^v \Rightarrow \ln y = v \cdot \ln u$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{v}{u} \cdot \frac{du}{dx} + \ln u \cdot \frac{dv}{dx}$$

$$\frac{dy}{dx} = y \left[\frac{v}{u} \cdot \frac{du}{dx} + \ln u \cdot \frac{dv}{dx} \right]$$

$$33) \quad \frac{d}{dx} u^v = u^v \cdot \left[\frac{v}{u} \cdot \frac{du}{dx} + \ln u \cdot \frac{dv}{dx} \right]$$

EX-18- Find $\frac{dy}{dx}$ for :

a) $y = x^{\cos x}$

b) $y = (\ln x + x)^{\tan x}$

Sol. -

$$a) \quad y = x^{\cos x} \Rightarrow \ln y = \cos x \cdot \ln x \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{\cos x}{x} + \ln x \cdot (-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y \left[\frac{\cos x}{x} - \sin x \cdot \ln x \right]$$

or by formula, where $u = x$ and $v = \cos x$

$$\frac{dy}{dx} = y \left[\frac{\cos x}{x} - \sin x \cdot \ln x \right]$$

$$\begin{aligned}
 b) \quad y &= (\ln x + x)^{\tan x} \Rightarrow \ln y = \tan x \cdot \ln(\ln x + x) \\
 &\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{\tan x}{\ln x + x} \cdot \left(\frac{1}{x} + 1\right) + \ln(\ln x + x) \cdot \sec^2 x \\
 &\Rightarrow \frac{dy}{dx} = y \left[\frac{(x+1) \cdot \tan x}{x(\ln x + x)} + \ln(\ln x + x) \cdot \sec^2 x \right]
 \end{aligned}$$

or by formula, where $u = \ln x + x$ and $v = \tan x$

$$\begin{aligned}
 \frac{dy}{dx} &= y \cdot \left[\frac{\tan x}{\ln x + x} \left(\frac{1}{x} + 1\right) + \ln(\ln x + x) \cdot \sec^2 x \right] \\
 &= y \cdot \left[\frac{(x+1) \cdot \tan x}{x(\ln x + x)} + \ln(\ln x + x) \cdot \sec^2 x \right]
 \end{aligned}$$

Problems -3

1. Find $\frac{dy}{dx}$ for the following functions :

- 1) $y = (x - 3)(1 - x)$ (ans.: $4 - 2x$)
- 2) $y = \frac{ax + b}{x}$ (ans.: $-\frac{b}{x^2}$)
- 3) $y = \frac{3x + 4}{2x + 3}$ (ans.: $\frac{1}{(2x + 3)^2}$)
- 4) $y = 3x^3 - 2\sqrt{x} + \frac{5}{x^2}$ (ans.: $9x^2 - \frac{1}{\sqrt{x}} - \frac{10}{x^3}$)
- 5) $y = \left(\sqrt{x^3} - \frac{1}{\sqrt{x^3}} \right)^2$ (ans.: $\frac{3(x^6 - 1)}{x^4}$)
- 6) $y = (2x - 1)^2 (3x + 2)^3 + \frac{1}{(x - 2)^2}$ (ans.: $(2x - 1)(3x + 2)^2 (30x - 1) - \frac{2}{(x - 2)^3}$)
- 7) $y = \ln(\ln x)$ (ans.: $\frac{1}{x \cdot \ln x}$)
- 8) $y = \ln(\cos x)$ (ans.: $-\tan x$)
- 9) $y = \sin x^3$ (ans.: $3x^2 \cdot \cos x^3$)
- 10) $y = \cos^{-3}(5x^2 + 2)$ (ans.: $\frac{30x \cdot \sin(5x^2 + 4)}{\cos^4(5x^2 + 4)}$)
- 11) $y = \tan x \cdot \sin x$ (ans.: $\sin x + \tan x \cdot \sec x$)
- 12) $y = \tan(\sec x)$ (ans.: $\sec^2(\sec x) \cdot \sec x \cdot \tan x$)
- 13) $y = \cot^3\left(\frac{x + 1}{x - 1}\right)$ (ans.: $\frac{6}{(x - 1)^2} \cdot \cot^2\left(\frac{x + 1}{x - 1}\right) \cdot \csc^2\left(\frac{x + 1}{x - 1}\right)$)
- 14) $y = \frac{\cos x}{x}$ (ans.: $-\frac{x \cdot \sin x + \cos x}{x^2}$)
- 15) $y = \sqrt{\tan \sqrt{2x + 7}}$ (ans.: $\frac{\sec^2 \sqrt{2x + 7}}{2\sqrt{2x + 7} \sqrt{\tan \sqrt{2x + 7}}}$)
- 16) $y = x^2 \cdot \sin x$ (ans.: $x^2 \cdot \cos x + 2x \cdot \sin x$)
- 17) $y = \csc^{\frac{2}{3}} \sqrt{5x}$ (ans.: $\frac{5}{3\sqrt{5x}} \cdot \frac{\cot \sqrt{5x}}{\csc^{\frac{2}{3}} \sqrt{5x}}$)
- 18) $y = x[\sin(\ln x) + \cos(\ln x)]$ (ans.: $2 \cdot \cos(\ln x)$)

- 19) $y = \text{Sin}^{-1}(5x^2)$ (ans.: $\frac{10x}{\sqrt{1-25x^4}}$)
- 20) $y = \text{Cot}^{-1}\left(\frac{1+x}{1-x}\right)$ (ans.: $-\frac{1}{1+x^2}$)
- 21) $y = \tan^{-1}\sqrt{4x^3-2}$ (ans.: $\frac{6x^2}{(4x^3-1)\sqrt{4x^3-2}}$)
- 22) $y = \text{Sec}^{-1}(3x^2+1)^3$ (ans.: $\frac{18x}{|3x^2+1|\sqrt{(3x^2+1)^6-1}}$)
- 23) $y = \text{Sin}^{-1}\frac{x^2}{2-x} + x^2.\text{Sec}^{-1}\frac{x}{2}$ (ans.: $\frac{4x-x^2}{(2-x)\sqrt{(2-x)^2-x^4}} + \frac{2x}{\sqrt{x^2-4}} + 2x.\text{Sec}^{-1}\frac{x}{2}$)
- 24) $y = \text{Sin}^{-1}2x.\text{Cos}^{-1}2x$ (ans.: $\frac{2(\text{Cos}^{-1}2x - \text{Sin}^{-1}2x)}{\sqrt{1-4x^2}}$)
- 25) $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$ (ans.: $\frac{y}{3}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right]$)
- 26) $y = \tan^{-1}(\ln x)$ (ans.: $\frac{1}{x(1+(\ln x)^2)}$)
- 27) $y^{\frac{4}{3}} = \frac{\sqrt{\sin x.\cos x}}{1+2.\ln x}$ (ans.: $\frac{3y}{4}\left(\frac{\cot x}{2} - \frac{\tan x}{2} - \frac{2}{x(1+2\ln x)}\right)$)
- 28) $\sqrt{y} = \frac{x^5.\tan^{-1}x}{(3-2x).\sqrt[3]{x}}$ (ans.: $2y\left(\frac{14}{3x} + \frac{1}{(1+x^2).\tan^{-1}x} + \frac{2}{3-2x}\right)$)
- 29) $y = \sec^{-1}e^{2x}$ (ans.: $\frac{2}{\sqrt{e^{4x}-1}}$)
- 30) $y = (\cos x)^{\sqrt{x}}$ (ans.: $\frac{y}{2\sqrt{x}}(\ln \cos x - 2x.\tan x)$)
- 31) $y = (\sin x)^{\tan x}$ (ans.: $y(1 + \sec^2 x.\ln \sin x)$)
- 32) $y = \sqrt{2x^2 + \cosh^2(5x)}$ (ans.: $\frac{2x + 5\cosh(5x).\sinh(5x)}{\sqrt{2x^2 + \cosh^2(5x)}}$)
- 33) $y = \sinh(\cos 2x)$ (ans.: $-2\sin 2x.\cosh(\cos 2x)$)
- 34) $y = \csc h\frac{1}{x}$ (ans.: $\frac{1}{x^2}.\csc h\frac{1}{x}.\coth\frac{1}{x}$)
- 35) $y = x^2.\tanh^2\sqrt{x}$ (ans.: $x.\tanh\sqrt{x}(\sqrt{x}\sec h^2\sqrt{x} + 2\tanh\sqrt{x})$)

$$\begin{aligned}
36) \quad y &= \ln \frac{\sin x \cdot \cos x + \tan^3 x}{\sqrt{x}} & (\text{ans.: } \frac{\cos^2 x - \sin^2 x + 3 \tan^2 x \cdot \sec^2 x}{\sin x \cdot \cos x + \tan^3 x} - \frac{1}{2x}) \\
37) \quad y &= \log_4 \sin x & (\text{ans.: } \frac{\cot x}{\ln 4}) \\
38) \quad y &= e^{(x^2 - e^{5x})} & (\text{ans.: } (2x - 5e^{5x})e^{(x^2 - e^{5x})}) \\
39) \quad y &= e^{x^2 \tan x} & (\text{ans.: } (x^2 \sec^2 x + 2x \tan x)e^{x^2 \tan x}) \\
40) \quad y &= 7^{\csc \sqrt{2x+3}} & (\text{ans.: } \frac{-7^{\csc \sqrt{2x+3}} \ln 7}{\sqrt{2x+3}} \csc \sqrt{2x+3} \cdot \cot \sqrt{2x+3}) \\
41) \quad y &= [\ln(x^2 + 2)]^2 \cos x & (\text{ans.: } \frac{4x \cdot \cos x}{x^2 + 2} - 2 \ln(x^2 + 2) \sin x) \\
42) \quad y &= \sinh^{-1}(\tan x) & (\text{ans.: } |\sec x|) \\
43) \quad y &= \sqrt{1 + (\ln x)^2} & (\text{ans.: } \frac{\ln x}{x \sqrt{1 + (\ln x)^2}}) \\
44) \quad y &= \frac{e^x}{\ln x} & (\text{ans.: } \frac{e^x (x \ln x - 1)}{x (\ln x)^2}) \\
45) \quad y &= x^3 \log_2(3 - 2x) & (\text{ans.: } 3x^2 \log_2(3 - 2x) - \frac{2x^3}{(3 - 2x) \ln 2}) \\
46) \quad y &= 2 \cosh^{-1} \frac{x}{2} + \frac{x}{2} \sqrt{x^2 - 4} & (\text{ans.: } \frac{x^2}{\sqrt{x^2 - 4}})
\end{aligned}$$

2. Verify the following derivatives :

$$\begin{aligned}
a) \quad \frac{d}{dx} \left[5x + \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 \right] &= 6 - \frac{1}{x^2} \\
b) \quad \frac{d}{dx} \left[\sqrt{x} (ax^2 + bx + c) \right] &= \frac{1}{2\sqrt{x}} (5ax^2 + 3bx + c)
\end{aligned}$$

3. Find the derivative of y with respect to x in the following functions :

$$\begin{aligned}
a) \quad y &= \frac{u^2}{u^2 + 1} \quad \text{and} \quad u = 3x^3 - 2 & (\text{ans.: } \frac{18x^2 y^2}{(3x^3 - 2)^3}) \\
b) \quad y &= \sqrt{u} + 2u \quad \text{and} \quad u = x^2 - 3 & (\text{ans.: } \frac{x}{\sqrt{x^2 - 3}} + 4x)
\end{aligned}$$

4. Find the second derivative for the following functions :

a) $y = \left(x + \frac{1}{x}\right)^3$ (ans.: $6x + \frac{6}{x^3} + \frac{12}{x^5}$)

b) $f(x) = \sqrt{2x} + \frac{2\sqrt{2}}{\sqrt{x}}$ at $x = 2$ (ans.: $\frac{1}{4}$)

c) $x^2 - 2xy + y^2 - 16x = 0$ (ans.: $\mp x^{-\frac{3}{2}}$)

5. Find the third derivative of the function :

$y = \sqrt{x^3}$ (ans.: $-\frac{3}{8y}$)

6. Show for $y = \frac{u}{v}$ that $y'' = \frac{v(vu'' - uv'') - 2v'(vu' - uv')}{v^3}$.

7. Show for $y = u.v$ that $y''' = uv''' + 3u'v'' + 3u''v' + u'''v$.

8. Show that $y = 35x^4 - 30x^2 + 3$ satisfies $(1 - x^2)y'' - 2xy' + 20y = 0$.

9. Find $\frac{dy}{dx}$ for the following implicit functions :

a) $x^3 + 4x\sqrt{y} - \frac{5y^2}{x} = 3$ (ans.: $\frac{3x^2 + 5y^2x^{-2} + 4\sqrt{y}}{10x^{-1}y - \frac{2x}{\sqrt{y}}}$)

b) $\sqrt{xy} + 1 = y$ (ans.: $\frac{y}{2\sqrt{xy} - x}$)

c) $3xy = (x^3 + y^3)^{\frac{3}{2}}$ (ans.: $\frac{3x^2\sqrt{x^3 + y^3} - 2y}{2x - 3y^2\sqrt{x^3 + y^3}}$)

d) $x^3 + x \cdot \tan^{-1} y = y$ (ans.: $\frac{(1 + y^2)(3x^2 + \tan^{-1} y)}{1 + y^2 - x}$)

e) $\sin^{-1}(xy) = \cos^{-1}(x - y)$ (ans.: $\frac{y\sqrt{1 - (x - y)^2} + \sqrt{1 - (xy)^2}}{\sqrt{1 - (xy)^2} - x\sqrt{1 - (x - y)^2}}$)

f) $y^2 \cdot \sin(xy) = \tan x$ (ans.: $\frac{\sec^2 x - y^3 \cdot \cos(xy)}{2y \cdot \sin(xy) + xy^2 \cdot \cos(xy)}$)

g) $\sinh y = \tan^2 x$ (ans.: $\frac{2 \cdot \tan x \cdot \sec^2 x}{\cosh y}$)

10. Prove the following formulas :

a) $\frac{d}{dx} \cot u = -\csc^2 u \cdot \frac{du}{dx}$

b) $\frac{d}{dx} \csc u = -\csc u \cdot \cot u \cdot \frac{du}{dx}$

c) $\frac{d}{dx} \cos^{-1} u = -\frac{1}{\sqrt{1 - u^2}} \cdot \frac{du}{dx}$

d) $\frac{d}{dx} \sec^{-1} u = \frac{1}{|u|\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$

e) $\frac{d}{dx} \sinh u = \cosh u \cdot \frac{du}{dx}$

f) $\frac{d}{dx} \csc h u = -\csc h u \cdot \coth u \cdot \frac{du}{dx}$

g) $\frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{1 + u^2}} \cdot \frac{du}{dx}$

h) $\frac{d}{dx} \sec h^{-1} u = -\frac{1}{|u|\sqrt{1 - u^2}} \cdot \frac{du}{dx}$

11. Show that the tangent to the hyperbola $x^2 - y^2 = 1$ at the point $P(\cosh u, \sinh u)$, cuts the x-axis at the point $(\operatorname{sech} u, 0)$ and except when vertical, cuts the y-axis at the point $(0, -\operatorname{csch} u)$.