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College of Sciences
Intelligent Medical Systems Department



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كلية العلوم
قسم الأنظمة الذكية

LECTURE: (4)

Subject: Applications of derivatives

Level: First

Lecturer: Dr. Mustafa Talal

Chapter four

Applications of derivatives

4-1- L'Hopital rule :

Suppose that $f(x_0) = g(x_0) = 0$ and that the functions f and g are both differentiable on an open interval (a, b) that contains the point x_0 . Suppose also that $g'(x) \neq 0$ at every point in (a, b) except possibly x_0 . Then :

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} \quad \text{provided the limit exists.}$$

Differentiate f and g as long as you still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = x_0$. Stop differentiating as soon as you get something else. L'Hopital's rule does not apply when either the numerator or denominator has a finite non-zero limit.

EX-1 – Evaluate the following limits :

$$\begin{array}{ll} 1) \lim_{x \rightarrow 0} \frac{\sin x}{x} & 2) \lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 5} - 3}{x^2 - 4} \\ 3) \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} & 4) \lim_{x \rightarrow \frac{\pi}{2}} -\left(x - \frac{\pi}{2}\right) \cdot \tan x \end{array}$$

Sol. –

$$\begin{array}{l} 1) \lim_{x \rightarrow 0} \frac{\sin x}{x} \Rightarrow \frac{0}{0} \text{ u sin g L'Hoptal's rule } \Rightarrow \\ \quad = \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1 \\ 2) \lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 5} - 3}{x^2 - 4} \Rightarrow \frac{0}{0} \text{ u sin g L'Hoptal's rule } \Rightarrow \\ \quad = \lim_{x \rightarrow 2} \frac{x}{2x} = \lim_{x \rightarrow 2} \frac{1}{2\sqrt{x^2 + 5}} = \frac{1}{2\sqrt{4 + 5}} = \frac{1}{6} \\ 3) \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \Rightarrow \frac{0}{0} \text{ u sin g L'Hoptal's rule } \Rightarrow \\ \quad = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \Rightarrow \frac{0}{0} \text{ u sin g L'Hoptal's rule } \Rightarrow \\ \quad = \frac{1}{6} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{1}{6} \end{array}$$

$$4) \lim_{x \rightarrow \frac{\pi}{2}} - (x - \frac{\pi}{2}) \tan x \Rightarrow 0 \cdot \infty \text{ we can't using L'Hopital's rule} \Rightarrow$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} - \frac{x - \frac{\pi}{2}}{\cos x} \cdot \lim_{x \rightarrow \frac{\pi}{2}} \sin x \Rightarrow \frac{0}{0} \text{ using L'Hopital's rule} \Rightarrow$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} - \frac{1}{-\sin x} \cdot \lim_{x \rightarrow \frac{\pi}{2}} \sin x = \frac{1}{\sin \frac{\pi}{2}} \cdot \sin \frac{\pi}{2} = 1$$

4-2- The slope of the curve :

Secant to the curve is a line through two points on a curve.

Slopes and tangent lines :

1. we start with what we can calculate , namely the slope of secant through P and a point Q nearby on the curve .
2. we find the limiting value of the secant slope (if it exists) as Q approaches p along the curve .
3. we take this number to be the slope of the curve at P and define the tangent to the curve at P to be the line through p with this slope .

The derivative of the function f is the slope of the curve :

$$\text{the slope} = m = f'(x) = \frac{dy}{dx}$$

EX-2- Write an equation for the tangent line at $x = 3$ of the curve :

$$f(x) = \frac{1}{\sqrt{2x+3}}$$

Sol.-

$$m = f'(x) = -\frac{1}{\sqrt{(2x+3)^3}} \Rightarrow [m]_{x=3} = f'(3) = -\frac{1}{27}$$

$$f(3) = \frac{1}{\sqrt{2*3+3}} = \frac{1}{3}$$

The equation of the tangent line is :

$$y - \frac{1}{3} = -\frac{1}{27}(x - 3) \Rightarrow 27y + x = 12$$

4-3- Velocity and acceleration and other rates of changes :

- The average velocity of a body moving along a line is :

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{\text{displacement}}{\text{time travelled}}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s = f(t)$ with respect to time t .

$$\text{i.e. } v = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t}$$

- The rate at which the particle's velocity increase is called its acceleration a . If a particle has an initial velocity v and a constant acceleration a , then its velocity after time t is $v + at$.

$$\text{average acceleration} = a_{av} = \frac{\Delta v}{\Delta t}$$

The acceleration at an instant is the limit of the average acceleration for an interval following that instant, as the interval tends to zero.

$$\text{i.e. } a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t}$$

- The average rate of a change in a function $y = f(x)$ over the interval from x to $x + \Delta x$ is :

$$\text{average rate of change} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The instantaneous rate of change of f at x is the derivative.

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ provided the limit exists.}$$

EX-3- The position s (in meters) of a moving body as a function of time t (in second) is : $s = 2t^2 + 5t - 3$; find :

- The displacement and average velocity for the time interval from $t = 0$ to $t = 2$ seconds .
- The body's velocity at $t = 2$ seconds .

Sol.-

$$a) \quad 1) \quad \Delta s = s(t + \Delta t) - s(t) = 2(t + \Delta t)^2 + 5(t + \Delta t) - 3 - [2t^2 + 5t - 3] \\ = (4t + 5)\Delta t + 2(\Delta t)^2$$

$$\text{at } t = 0 \text{ and } \Delta t = 2 \Rightarrow \Delta s = (4 * 0 + 5) * 2 + 2 * 2^2 = 18$$

$$2) \quad v_{av} = \frac{\Delta s}{\Delta t} = \frac{(4t + 5)\Delta t + 2(\Delta t)^2}{\Delta t} = 4t + 5 + 2\Delta t$$

$$\text{at } t = 0 \text{ and } \Delta t = 2 \Rightarrow v_{av} = 4 * 0 + 5 + 2 * 2 = 9$$

$$b) \quad v(t) = \frac{d}{dt} f(t) = 4t + 5$$

$$v(2) = 4 * 2 + 5 = 13$$

EX-4- A particle moves along a straight line so that after t (seconds), its distance from O a fixed point on the line is s (meters), where $s = t^3 - 3t^2 + 2t$:

i) when is the particle at O ?

ii) what is its velocity and acceleration at these times ?

iii) what is its average velocity during the first second ?

iv) what is its average acceleration between $t = 0$ and $t = 2$?

Sol. -

$$i) \quad \text{at } s = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t - 1)(t - 2) = 0 \\ \text{either } t = 0 \text{ or } t = 1 \text{ or } t = 2 \text{ sec.}$$

$$ii) \quad \text{velocity} = v(t) = 3t^2 - 6t + 2 \Rightarrow v(0) = 2 \text{ m / s} \\ \Rightarrow v(1) = -1 \text{ m / s} \\ \Rightarrow v(2) = 2 \text{ m / s}$$

$$\text{acceleration} = a(t) = 6t - 6 \Rightarrow a(0) = -6 \text{ m / s}^2 \\ \Rightarrow a(1) = 0 \text{ m / s}^2 \\ \Rightarrow a(2) = 6 \text{ m / s}^2$$

$$iii) \quad v_{av} = \frac{\Delta s}{\Delta t} = \frac{s(1) - s(0)}{1 - 0} = \frac{1 - 3 + 2 - 0}{1} = 0 \text{ m / s}$$

$$iv) \quad a_{av} = \frac{\Delta v}{\Delta t} = \frac{v(2) - v(0)}{2 - 0} = \frac{2 - 2}{2} = 0 \text{ m / s}^2$$

4-4- Maxima and Minima :

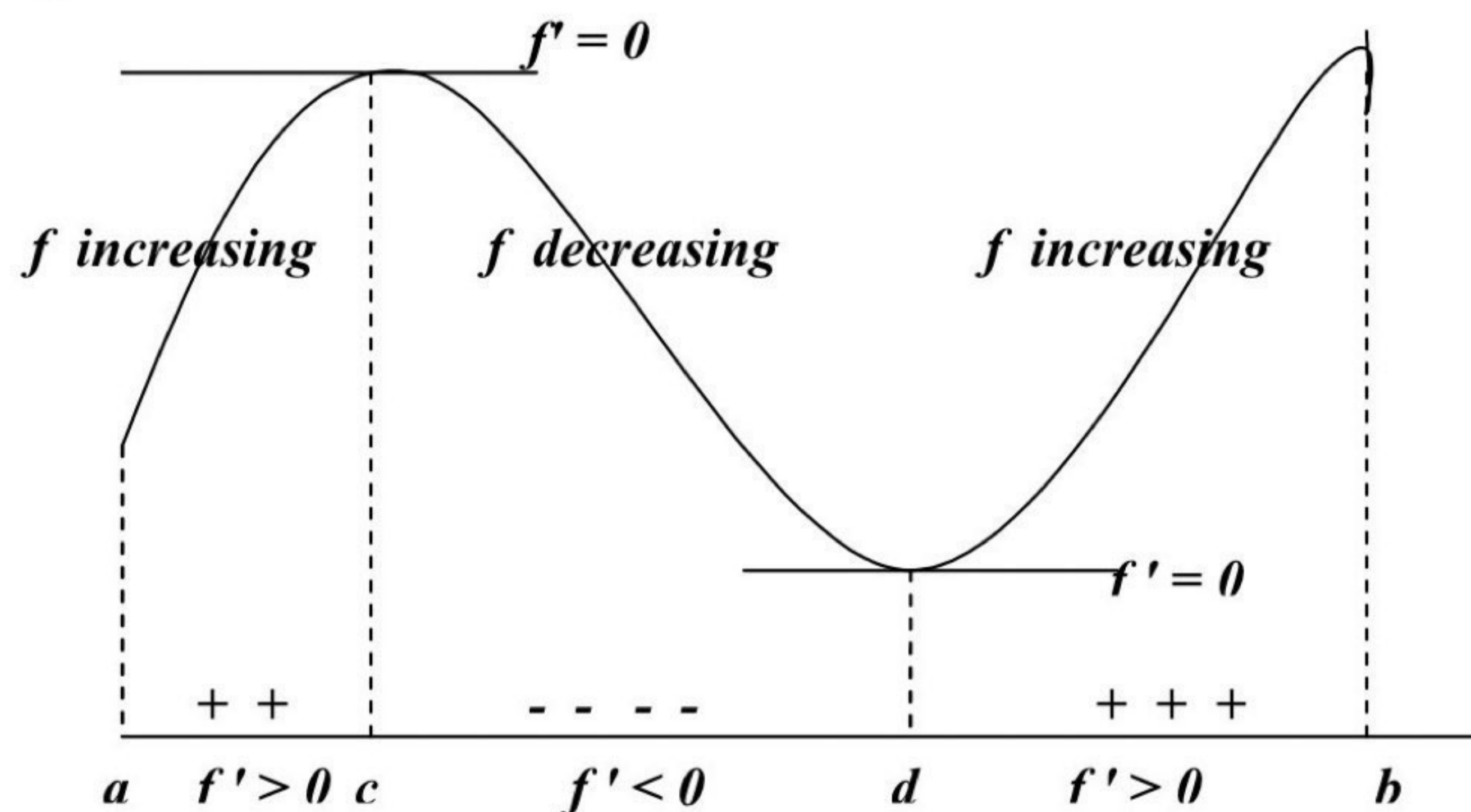
Increasing and decreasing function : Let f be defined on an interval and x_1, x_2 denoted a number on that interval :

- If $f(x_1) < f(x_2)$ when ever $x_1 < x_2$ then f is increasing on that interval .
- If $f(x_1) > f(x_2)$ when ever $x_1 < x_2$ then f is decreasing on that interval .
- If $f(x_1) = f(x_2)$ for all values of x_1, x_2 then f is constant on that interval .

The first derivative test for rise and fall : Suppose that a function f has a derivative at every point x of an interval I . Then :

- f increases on I if $f'(x) > 0, \forall x \in I$
- f decreases on I if $f'(x) < 0, \forall x \in I$

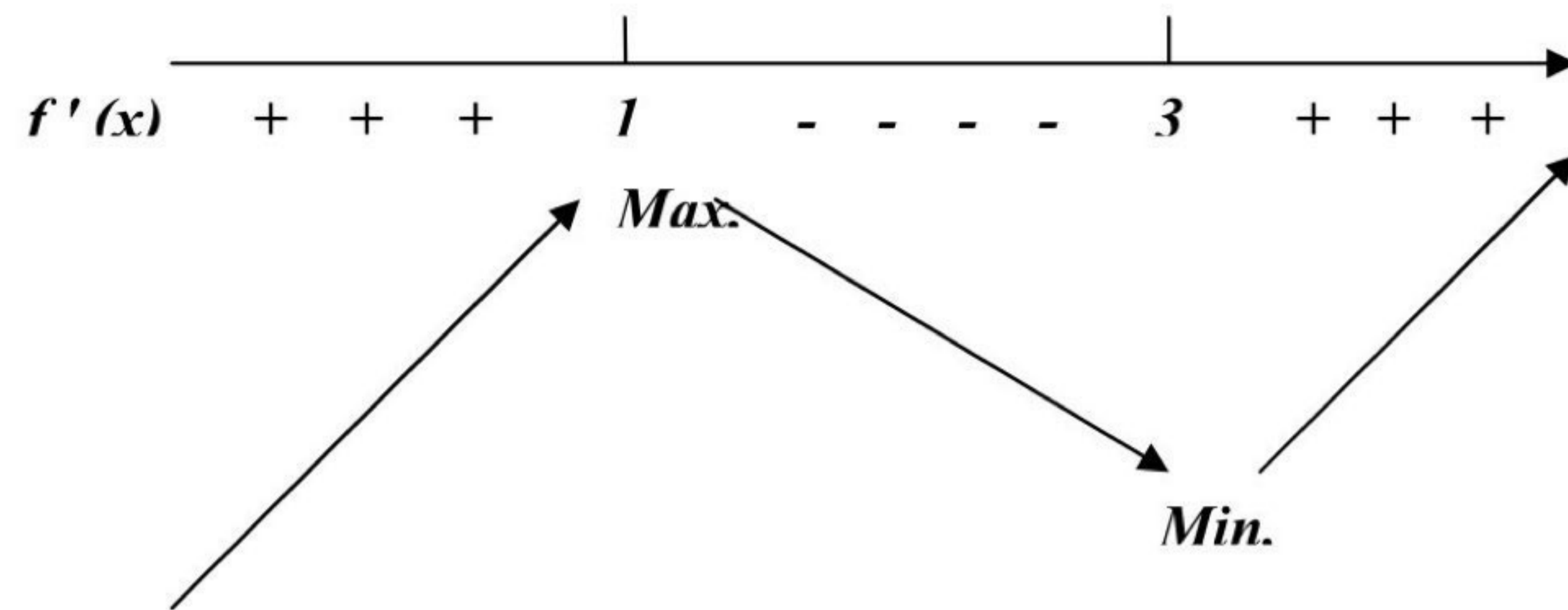
If f' changes from positive to negative values as x passes from left to right through a point c , then the value of f at c is a local maximum value of f , as shown in below figure . That is $f(c)$ is the largest value the function takes in the immediate neighborhood at $x = c$.



Similarly, if f' changes from negative to positive values as x passes left to right through a point d , then the value of f at d is a local minimum value of f . That is $f(d)$ is the smallest value of f takes in the immediate neighborhood of d .

EX-5 – Graph the function : $y = f(x) = \frac{x^3}{3} - 2x^2 + 3x + 2$.

Sol.- $f'(x) = x^2 - 4x + 3 \Rightarrow (x - 1)(x - 3) = 0 \Rightarrow x = 1, 3$

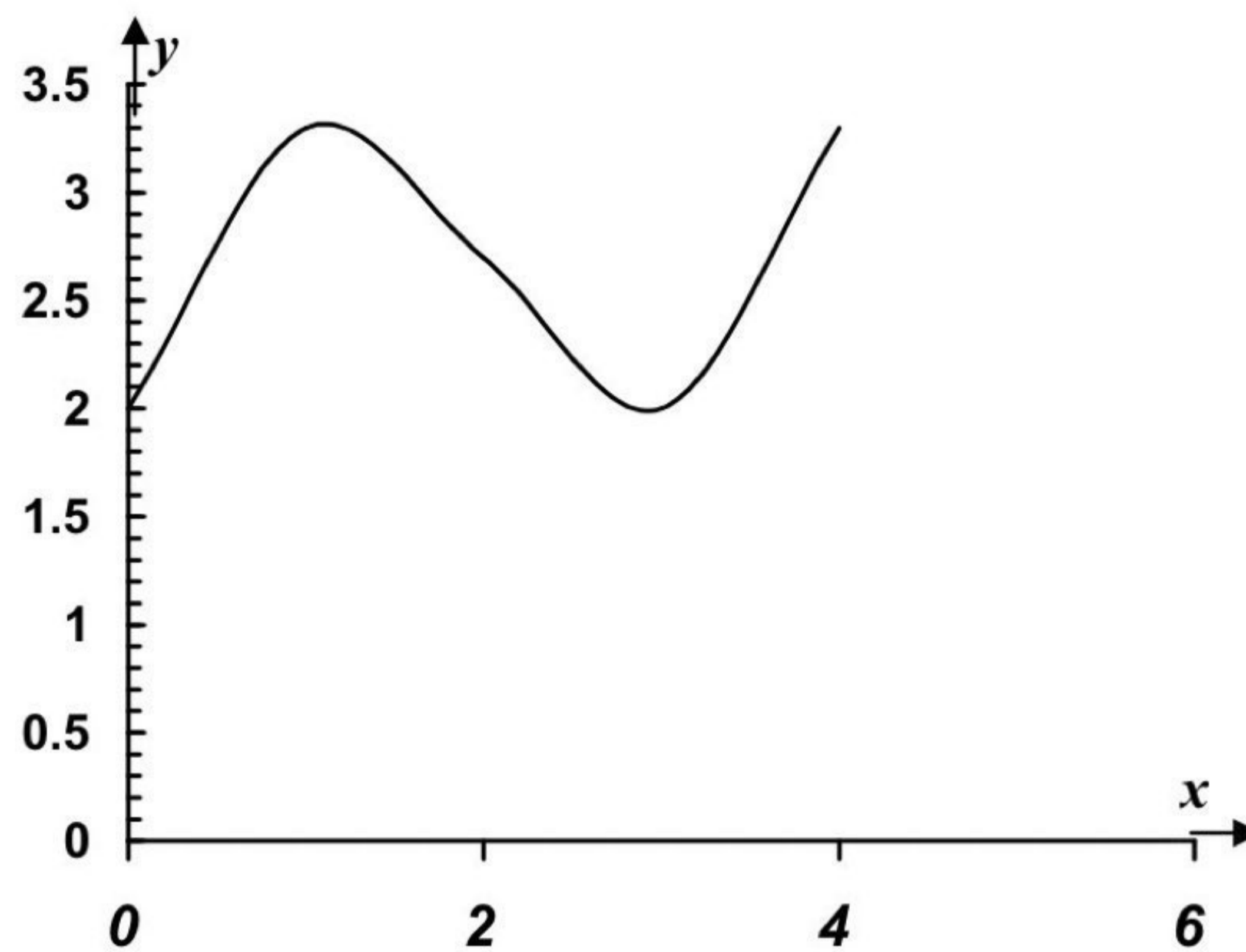


The function has a local maximum at $x = 1$ and a local minimum at $x = 3$.

To get a more accurate curve, we take :

x	0	1	2	3	4
$f(x)$	2	3.3	2.7	2	3.3

Then the graph of the function is :



Concave down and concave up : The graph of a differentiable function $y = f(x)$ is concave down on an interval where f' decreases, and concave up on an interval where f' increases.

The second derivative test for concavity : The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$, concave up on any interval where $y'' > 0$.

Point of inflection : A point on the curve where the concavity changes is called a point of inflection. Thus, a point of inflection on a twice-differentiable curve is a point where y'' is positive on one side and negative on other, i.e. $y'' = 0$.

EX-6 – Sketch the curve : $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$.

Sol. -

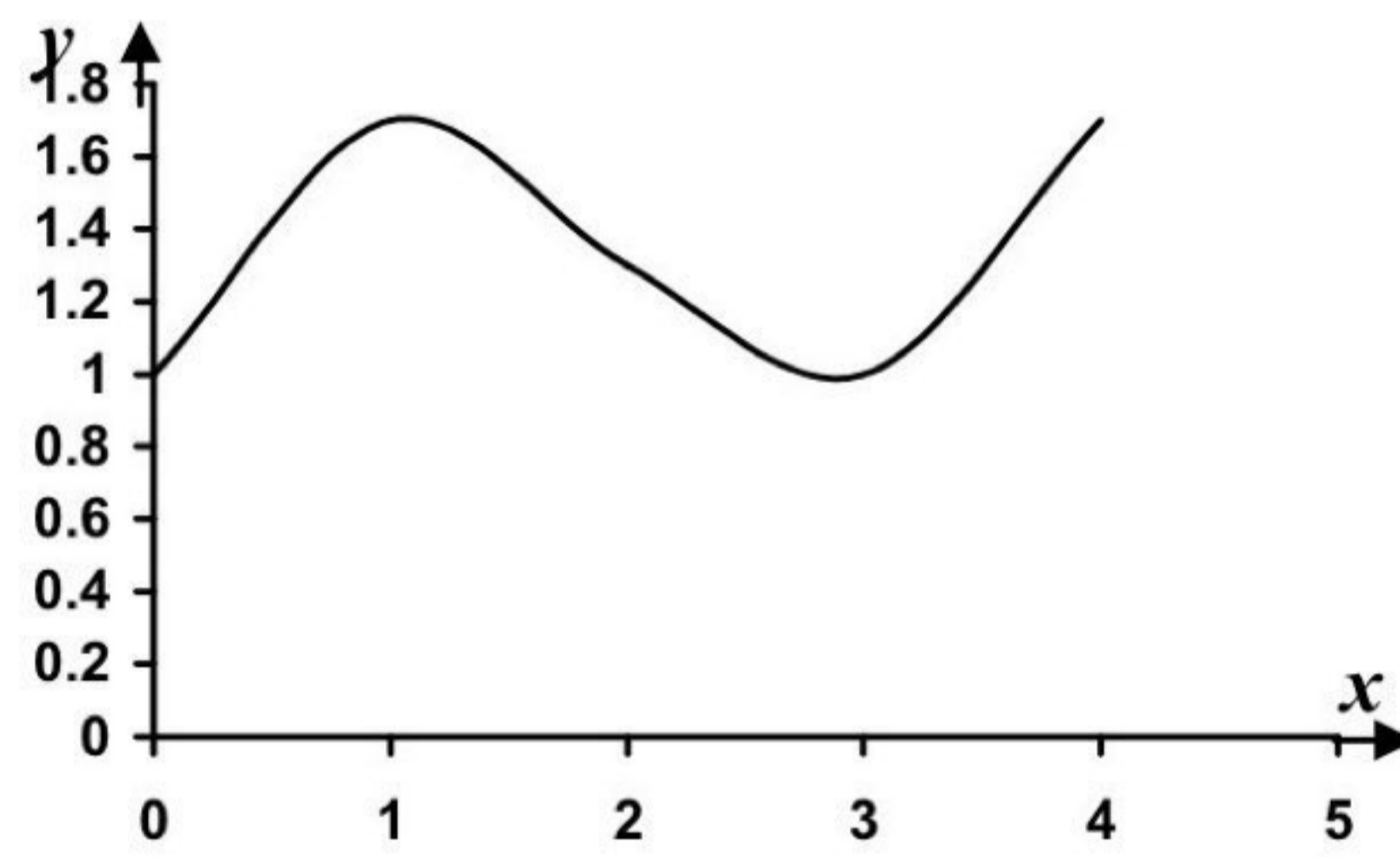
$$y' = \frac{1}{2}x^2 - 2x + \frac{3}{2} = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1, 3$$

$$y'' = x - 2 \Rightarrow \text{at } x = 1 \Rightarrow y'' = 1 - 2 = -1 < 0 \text{ concave down .}$$

$$\Rightarrow \text{at } x = 3 \Rightarrow y'' = 3 - 2 > 0 \text{ concave up .}$$

$$\Rightarrow \text{at } y'' = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2 \text{ point of inflection .}$$

x	0	1	2	3	4
y	1	1.7	1.3	1	1.7



EX-7 – What value of a makes the function :

$$f(x) = x^2 + \frac{a}{x}, \text{ have :}$$

- i) a local minimum at $x = 2$?
- ii) a local minimum at $x = -3$?
- iii) a point of inflection at $x = 1$?
- iv) show that the function can't have a local maximum for any value of a .

Sol. -

$$f(x) = x^2 + \frac{a}{x} \Rightarrow \frac{df}{dx} = 2x - \frac{a}{x^2} = 0 \Rightarrow a = 2x^3 \text{ and } \frac{d^2y}{dx^2} = 2 + \frac{2a}{x^3}$$

- i) at $x = 2 \Rightarrow a = 2 * 8 = 16$ and $\frac{d^2 f}{dx^2} = 2 + \frac{2 * 16}{2^3} = 6 > 0$ Mini.
- ii) at $x = -3 \Rightarrow a = 2(-3)^3 = -54$ and $\frac{d^2 f}{dx^2} = 2 + \frac{2(-54)}{(-3)^3} = 6 > 0$ Mini.
- iii) at $x = 1 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2a}{1} = 0 \Rightarrow a = -1$
- iv) $a = 2x^3 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2(2x^3)}{x^3} = 6 > 0$

Since $\frac{d^2 f}{dx^2} > 0$ for all value of x in $a = 2x^3$.

Hence the function don't have a local maximum .

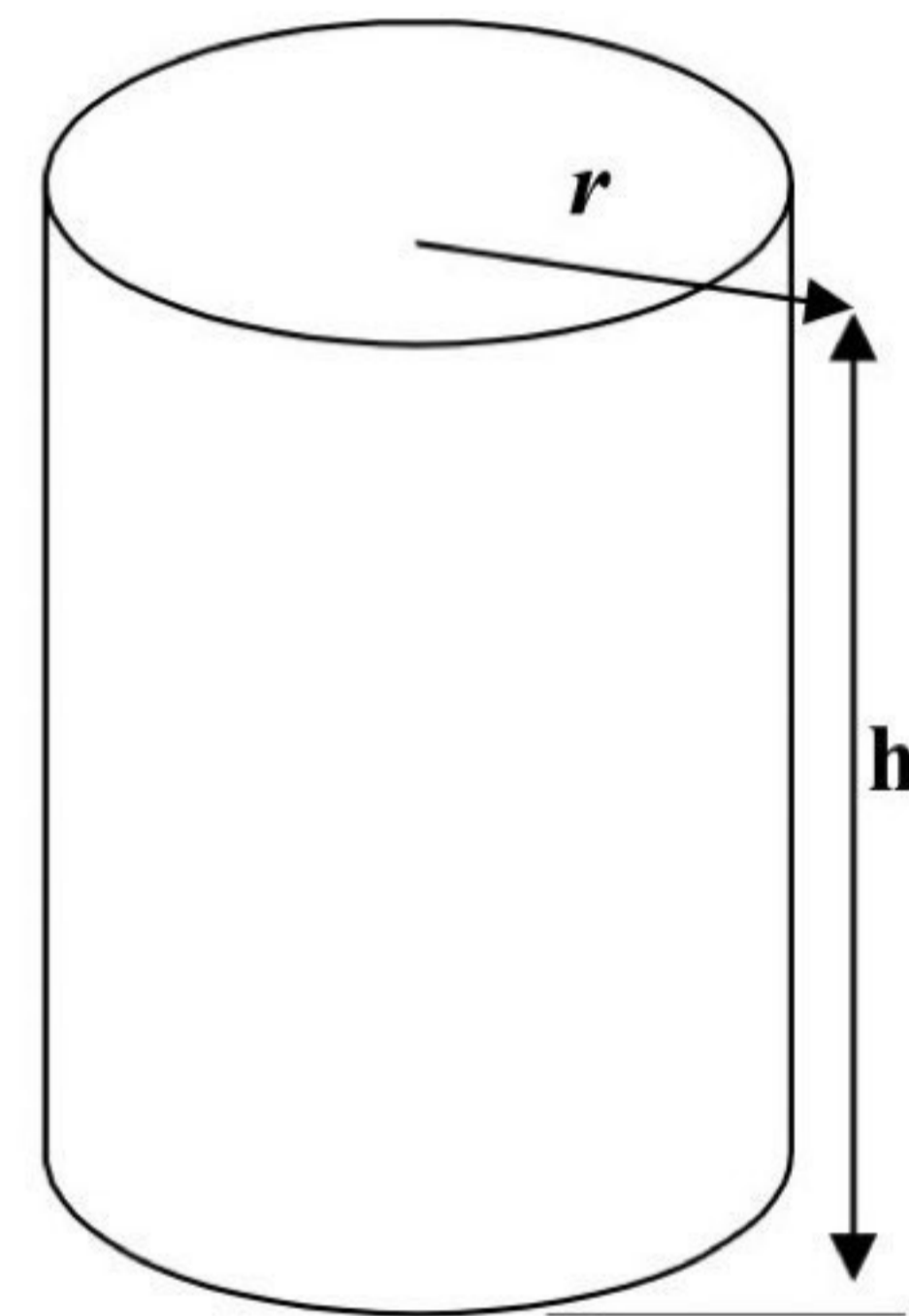
EX-8 – What are the best dimensions (use the least material) for a tin can which is to be in the form of a right circular cylinder and is to hold 1 gallon (231 cubic inches) ?

Sol. – The volume of the can is :

$$v = \pi r^2 h = 231 \Rightarrow h = \frac{231}{\pi r^2}$$

where r is radius , h is height .

The total area of the outer surface (top, bottom , and side) is :



$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{231}{\pi r^2} \Rightarrow A = 2\pi r^2 + \frac{462}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{462}{r^2} = 0 \Rightarrow r = 3.3252 \text{ inches}$$

$$\frac{d^2 A}{dr^2} = 4\pi + \frac{924}{r^3} = 4\pi + \frac{924}{(3.3252)^3} = 37.714 > 0 \Rightarrow \text{min.}$$

$$h = \frac{231}{\pi r^2} = \frac{231}{\frac{22}{7} (3.3252)^2} = 6.6474 \text{ inches}$$

The dimensions of the can of volume 1 gallon have minimum surface area are :

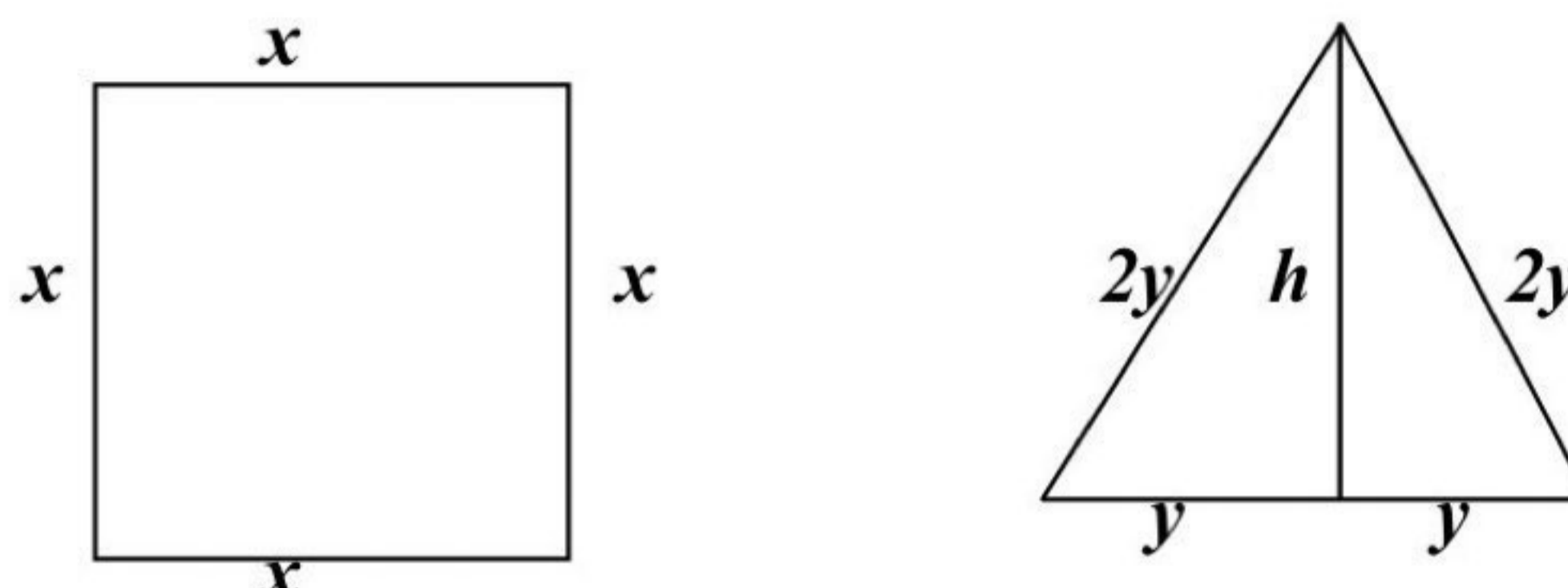
$$r = 3.3252 \text{ in. and } h = 6.6474 \text{ in.}$$

EX-9 – A wire of length L is cut into two pieces, one being bent to form a square and the other to form an equilateral triangle. How should the wire be cut:

- if the sum of the two areas is minimum.
- if the sum of the two areas is maximum.

Sol. : Let x is a length of square.

$2y$ is the edge of triangle.



The perimeter is $p = 4x + 6y = L \Rightarrow x = \frac{1}{4}(L - 6y)$.

$(2y)^2 = y^2 + h^2 \Rightarrow h = \sqrt{3}y$ from triangle.

The total area is $A = x^2 + yh = \frac{1}{16}(L - 6y)^2 + y\sqrt{3}y$
 $\Rightarrow A = \frac{1}{16}(L - 6y)^2 + \sqrt{3}y^2$

$$\frac{dA}{dy} = \frac{-3}{4}(L - 6y) + 2\sqrt{3}y = 0 \Rightarrow y = \frac{3L}{18 + 8\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9}{2} + 2\sqrt{3} > 0 \Rightarrow \text{min.}$$

- To minimized total areas cut for triangle $6y = \frac{9L}{9 + 4\sqrt{3}}$

And for square $L - \frac{9L}{9 + 4\sqrt{3}} = \frac{4\sqrt{3}L}{9 + 4\sqrt{3}}$.

- To maximized the value of A on endpoints of the interval

$$0 \leq 4x \leq L \Rightarrow 0 \leq x \leq \frac{L}{4}$$

$$\text{at } x = 0 \Rightarrow y = \frac{L}{6} \text{ and } h = \frac{L}{2\sqrt{3}} \Rightarrow A_1 = \frac{L^2}{12\sqrt{3}}$$

$$\text{at } x = \frac{L}{4} \Rightarrow y = 0 \Rightarrow A_2 = \frac{L^2}{16}$$

$$\text{Since } A_2 = \frac{L^2}{16} > A_1 = \frac{L^2}{12\sqrt{3}}$$

Hence the wire should not be cut at all but should be bent into a square.

Problems – 4

1. Find the velocity v if a particle's position at time t is $s = 180t - 16t^2$
When does the velocity vanish ? (ans.: 5.625)

2. If a ball is thrown straight up with a velocity of 32 ft./sec. , its high after t sec. is given by the equation $s = 32t - 16t^2$. At what instant will the ball be at its highest point ? and how high will it rise ?
(ans.: 1, 16)

3. A stone is thrown vertically upwards at 35 m./sec. . Its height is :
 $s = 35t - 4.9t^2$ in meter above the point of projection where t is time in second later :
 - a) What is the distance moved, and the average velocity during the 3rd sec. (from $t = 2$ to $t = 3$) ?
 - b) Find the average velocity for the intervals $t = 2$ to $t = 2.5$, $t = 2$ to $t = 2.1$; $t = 2$ to $t = 2 + h$.
 - c) Deduce the actual velocity at the end of the 2nd sec. .
(ans.: a) 10.5 , 10.5 ; b) 12.95, 14.91, 15.4-4.9h , c) 15.4)

4. A stone is thrown vertically upwards at 24.5 m./sec. from a point on the level with but just beyond a cliff ledge . Its height above the ledge t sec. later is $4.9t (5 - t)$ m. . If its velocity is v m./sec. , differentiate to find v in terms of t :
 - i) when is the stone at the ledge level ?
 - ii) find its height and velocity after 1 , 2 , 3 , and 6 sec. .
 - iii) what meaning is attached to negative value of s ? a negative value of v ?
 - iv) when is the stone momentarily at rest ? what is the greatest height reached ?
 - v) find the total distance moved during the 3rd sec. .
(ans.: $v=24.5-9.8t$; i)0,5; ii)19.6,29.4,29.4,-29.4;14.7,4.9, -4.9,-34.3; iv)2.5;30.625; v)2.45)

5. A stone is thrown vertically downwards with a velocity of 10 m./sec. , and gravity produces on it an acceleration of 9.8 m./sec.² :
 - a) what is the velocity after 1 , 2 , 3 , t sec. ?
 - b) sketch the velocity –time graph . (ans.: 19.8, 29.6, 39.4, 10+9.8t)

6. A car accelerates from 5 km./h. to 41 km./h. in 10 sec. . Express this acceleration in : i)km./h. per sec. ii) m./sec.², iii) km./h.² .
(ans.: i)3.6; ii)1; iii) 12960)

7. A car can accelerate at 4 m./sec.^2 . How long will it take to reach 90 km./h. from rest ? *(ans.: 6.25)*
8. An express train reducing its velocity to 40 km./h. , has to apply the brakes for 50 sec. . If the retardation produced is 0.5 m./sec.^2 , find its initial velocity in km./h. . *(ans.: 130)*
9. At the instant from which time is measured a particle is passing through O and traveling towards A , along the straight line OA . It is $s \text{ m.}$ from O after $t \text{ sec.}$ where $s = t(t - 2)^2$:
- i) when is it again at O ?
 - ii) when and where is it momentarily at rest ?
 - iii) what is the particle's greatest displacement from O , and how far does it moves , during the first 2 sec. ?
 - iv) what is the average velocity during the 3^{rd} sec. ?
 - v) at the end of the 1^{st} sec. where is the particle, which way is it going , and is its speed increasing or decreasing ?
 - vi) repeat (v) for the instant when $t = -1$.
- (ans.: i)2; ii)0,32/27; iii)64/27; iv)3; v)OA; inceasing; vi)AO; decreasing)*
10. A particle moves in a straight line so that after $t \text{ sec.}$ it is $s \text{ m.}$, from a fixed point O on the line , where $s = t^4 + 3t^2$. Find :
- i) The acceleration when $t = 1$, $t = 2$, and $t = 3$.
 - ii) The average acceleration between $t = 1$ and $t = 3$.
- (ans.: i)18, 54,114; ii)58)*
11. A particle moves along the x-axis in such away that its distance $x \text{ cm.}$ from the origin after $t \text{ sec.}$ is given by the formula $x = 27t - 2t^2$ what are its velocity and acceleration after 6.75 sec. ? How long does it take for the velocity to be reduced from 15 cm./sec. to 9 cm./sec. , and how far does the particle travel mean while ? *(ans.: 0,-4,1.5 ;18)*
12. A point moves along a straight line OX so that its distance $x \text{ cm.}$ from the point O at time $t \text{ sec.}$ is given by the formula $x = t^3 - 6t^2 + 9t$. Find :
- i) at what times and in what positions the point will have zero velocity .
 - ii) its acceleration at these instants .
 - iii) its velocity when its acceleration is zero .
- (ans.: i)1,3;4,0; ii)-6,6; iii)-3)*

13. A particle moves in a straight line so that its distance x cm. from a fixed point O on the line is given by $x = 9t^2 - 2t^3$ where t is the time in seconds measured from O . Find the speed of the particle when $t = 3$. Also find the distance from O of the particle when $t = 4$, and show that it is then moving towards O . (ans.: 0, 16)

14. Find the limits for the following functions by using L'Hopital's rule :

$$1) \lim_{x \rightarrow \infty} \frac{5x^2 - 3x}{7x^2 + 1}$$

$$2) \lim_{t \rightarrow 0} \frac{\sin t^2}{t}$$

$$3) \lim_{x \rightarrow \frac{\pi}{2}} \frac{2x - \pi}{\cos x}$$

$$4) \lim_{t \rightarrow 0} \frac{\cos t - 1}{t^2}$$

$$5) \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{1 + \cos 2x}$$

$$6) \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

$$7) \lim_{x \rightarrow 1} \frac{2x^2 - (3x + 1)\sqrt{x} + 2}{x - 1}$$

$$8) \lim_{x \rightarrow 0} \frac{x(\cos x - 1)}{\sin x - x}$$

$$9) \lim_{x \rightarrow 0} x \cdot \csc^2 \sqrt{2x}$$

$$10) \lim_{x \rightarrow 0} \frac{\sin x^2}{x \cdot \sin x}$$

$$(ans. : 1) \frac{5}{7}; 2) 0; 3) -2; 4) -\frac{1}{2}; 5) \frac{1}{4}; 6) \sqrt{2}; 7) -1; 8) 3; 9) \frac{1}{2}; 10) 1)$$

15. Find any local maximum and local minimum values, then sketch each curve by using first derivative :

$$1) f(x) = x^3 - 4x^2 + 4x + 5 \quad (ans. : max.(0.7, 6.2); min.(2, 5))$$

$$2) f(x) = \frac{x^2 - 1}{x^2 + 1} \quad (ans. : min.(0, -1))$$

$$3) f(x) = x^5 - 5x - 6 \quad (ans. : max.(-1, -2); min.(1, -10))$$

$$4) f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}} \quad (ans. : min.(0.25, -0.47))$$

16. Find the interval of x -values on which the curve is concave up and concave down, then sketch the curve :

$$1) f(x) = \frac{x^3}{3} + x^2 - 3x \quad (ans. : up(-1, \infty); down(-\infty, -1))$$

$$2) f(x) = x^2 - 5x + 6 \quad (ans. : up(-\infty, \infty))$$

$$3) f(x) = x^3 - 2x^2 + 1 \quad (ans. : up(\frac{2}{3}, \infty); down(-\infty, \frac{2}{3}))$$

$$4) f(x) = x^4 - 2x^2 \quad (ans. : up(-\infty, -\frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \infty); down(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}))$$

17. Sketch the following curve by using second derivative :

1) $y = \frac{x}{1+x^2}$ (ans. : max.(1,0.5); min.(-1,-0.5))

2) $y = -x(x-7)^2$ (ans. : max.(7,0); min.(2.3,-50.8))

3) $y = (x+2)^2(x-3)$ (ans. : max.(-2,0); min.(1.3,-18.5))

4) $y = x^2(5-x)$ (ans. : max.(3.3,18.5); min.(0,0))

18. What is the smallest perimeter possible for a rectangle of area 16 in.² ? (ans.: 16)

19. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the parabola $y = 12 - x^2$. (ans.:32)

20) A rectangular plot is to be bounded on one side by a straight river and enclosed on the other three sides by a fence . With 800 m. of fence at your disposal . What is the largest area you can enclose ? (ans.:80000)

21) Show that the rectangle that has maximum area for a given perimeter is a square .

22) A wire of length L is available for making a circle and a square . How should the wire be divided between the two shapes to maximize the sum of the enclosed areas? (ans.: all bent into a circle)

23) A closed container is made from a right circular cylinder of radius r and height h with a hemispherical dome on top . Find the relationship between r and h that maximizes the volume for a given surface area s . (ans. : $r = h = \sqrt{\frac{s}{5\pi}}$)

24) An open rectangular box is to be made from a piece of cardboard 8 in. wide and 15 in. long by cutting a square from each corner and bending up the sides Find the dimensions of the box of largest volume . (ans.: height=5/3; width=14/3; length=35/3)

Chapter five

Integration

5-1- Indefinite integrals :

The set of all anti derivatives of a function is called indefinite integral of the function.

Assume u and v denote differentiable functions of x , and a , n , and c are constants, then the integration formulas are:-

$$1) \int du = u(x) + c$$

$$2) \int a \cdot u(x) dx = a \int u(x) dx$$

$$3) \int (u(x) \mp v(x)) dx = \int u(x) dx \mp \int v(x) dx$$

$$4) \int u^n du = \frac{u^{n+1}}{n+1} + c \quad \text{when } n \neq -1 \quad \& \quad \int u^{-1} du = \int \frac{1}{u} du = \ln u + c$$

$$5) \int a^u du = \frac{a^u}{\ln a} + c \quad \Rightarrow \quad \int e^u du = e^u + c$$

EX-1 – Evaluate the following integrals:

$$1) \int 3x^2 dx$$

$$6) \int \frac{x+3}{\sqrt{x^2+6x}} dx$$

$$2) \int \left(\frac{1}{x^2} + x \right) dx$$

$$7) \int \frac{x+2}{x^2} dx$$

$$3) \int x\sqrt{x^2+1} dx$$

$$8) \int \frac{e^x}{1+3e^x} dx$$

$$4) \int (2t + t^{-1})^2 dt$$

$$9) \int 3x^3 \cdot e^{-2x^4} dx$$

$$5) \int \sqrt{(z^2 - z^{-2})^2 + 4} dz$$

$$10) \int 2^{-4x} dx$$

Sol. –

$$1) \int 3x^2 dx = 3 \int x^2 dx = 3 \frac{x^3}{3} + c = x^3 + c$$

$$2) \int (x^{-2} + x) dx = \int x^{-2} dx + \int x dx = \frac{x^{-1}}{-1} + \frac{x^2}{2} + c = -\frac{1}{x} + \frac{x^2}{2} + c$$

$$3) \int x\sqrt{x^2 + 1} dx = \frac{1}{2} \int 2x(x^2 + 1)^{1/2} dx = \frac{1}{2} \frac{(x^2 + 1)^{3/2}}{3/2} + c = \frac{1}{3} \sqrt{(x^2 + 1)^3} + c$$

$$4) \int (2t + t^{-1})^2 dt = \int (4t^2 + 4 + t^{-2}) dt = 4 \frac{t^3}{3} + 4t + \frac{t^{-1}}{-1} + c = \frac{4}{3} t^3 + 4t - \frac{1}{t} + c$$

$$5) \int \sqrt{(z^2 - z^{-2})^2 + 4} dz = \int \sqrt{z^4 - 2 + z^{-4} + 4} dz = \int \sqrt{z^4 + 2 + z^{-4}} dz$$

$$= \int \sqrt{(z^2 + z^{-2})^2} dz = \int (z^2 + z^{-2}) dz = \frac{z^3}{3} + \frac{z^{-1}}{-1} + c = \frac{1}{3} z^3 - \frac{1}{z} + c$$

$$6) \int \frac{x+3}{\sqrt{x^2+6x}} dx = \frac{1}{2} \int (2x+6) \cdot (x^2+6x)^{-1/2} dx$$

$$= \frac{1}{2} \cdot \frac{(x^2+6x)^{1/2}}{1/2} + c = \sqrt{x^2+6x} + c$$

$$7) \int \frac{x+2}{x^2} dx = \int \left(\frac{x}{x^2} + \frac{2}{x^2} \right) dx = \int (x^{-1} + 2x^{-2}) dx = \ln x + \frac{2x^{-1}}{-1} + c = \ln x - \frac{2}{x} + c$$

$$8) \int \frac{e^x}{1+3e^x} dx = \frac{1}{3} \int 3e^x (1+3e^x)^{-1} dx = \frac{1}{3} \ln(1+3e^x) + c$$

$$9) \int 3x^3 \cdot e^{-2x^4} dx = -\frac{3}{8} \int -8x^3 \cdot e^{-2x^4} dx = -\frac{3}{8} \cdot e^{-2x^4} + c$$

$$10) \int 2^{-4x} dx = -\frac{1}{4} \int 2^{-4x} \cdot (-4 dx) = -\frac{1}{4} \cdot 2^{-4x} \cdot \frac{1}{\ln 2} + c$$

5-2- Integrals of trigonometric functions :

The integration formulas for the trigonometric functions are:

$$6) \int \sin u \cdot du = -\cos u + c$$

$$7) \int \cos u \cdot du = \sin u + c$$

$$8) \int \tan u \cdot du = -\ln|\cos u| + c$$

$$9) \int \cot u \cdot du = \ln|\sin u| + c$$

$$10) \int \sec u \cdot du = \ln|\sec u + \tan u| + c$$

$$11) \int \csc u \cdot du = -\ln|\csc u + \cot u| + c$$

$$12) \int \sec^2 u \cdot du = \tan u + c$$

$$13) \int \csc^2 u \cdot du = -\cot u + c$$

$$14) \int \sec u \cdot \tan u \cdot du = \sec u + c$$

$$15) \int \csc u \cdot \cot u \cdot du = -\csc u + c$$

EX-2- Evaluate the following integrals:

$$1) \int \cos(3\theta - 1) d\theta$$

$$2) \int x \cdot \sin(2x^2) dx$$

$$3) \int \cos^2(2y) \cdot \sin(2y) dy$$

$$4) \int \sec^3 x \cdot \tan x dx$$

$$5) \int \sqrt{2 + \sin 3t} \cdot \cos 3t dt$$

$$6) \int \frac{d\theta}{\cos^2 \theta}$$

$$7) \int (1 - \sin^2 3t) \cdot \cos 3t dt$$

$$8) \int \tan^3(5x) \cdot \sec^2(5x) dx$$

$$9) \int \sin^4 x \cdot \cos^3 x dx$$

$$10) \int \frac{\cot^2 \sqrt{x}}{\sqrt{x}} dx$$

Sol.-

$$1) \frac{1}{3} \int 3 \cos(3\theta - 1) d\theta = \frac{1}{3} \sin(3\theta - 1) + c$$

$$2) \frac{1}{4} \int 4x \cdot \sin(2x^2) dx = -\frac{1}{4} \cos(2x^2) + c$$

$$3) -\frac{1}{2} \int (\cos 2y)^2 \cdot (-2 \sin 2y dy) = -\frac{1}{2} \cdot \frac{(\cos 2y)^3}{3} + c = -\frac{1}{6} (\cos 2y)^3 + c$$

$$4) \int \sec^2 x \cdot (\sec x \cdot \tan x \cdot dx) = \frac{\sec^3 x}{3} + c$$

$$5) \frac{1}{3} \int (2 + \sin 3t)^{1/2} (3 \cos 3t dt) = \frac{1}{3} \cdot \frac{(2 + \sin 3t)^{3/2}}{3/2} + c = \frac{2}{9} \sqrt{(2 + \sin 3t)^3} + c$$

$$6) \int \frac{d\theta}{\cos^2 \theta} = \int \sec^2 \theta \cdot d\theta = \tan \theta + c$$

$$7) \int (1 - \sin^2 3t) \cdot \cos 3t dt = \frac{1}{3} \int 3 \cos 3t dt - \frac{1}{3} \int (\sin 3t)^2 \cdot 3 \cos 3t dt$$

$$= \frac{1}{3} \sin 3t - \frac{1}{3} \cdot \frac{\sin^3 3t}{3} + c = \frac{1}{3} \sin 3t - \frac{1}{9} \sin^3 3t + c$$

$$8) \frac{1}{5} \int \tan^3 5x \cdot (5 \sec^2 5x dx) = \frac{1}{5} \cdot \frac{\tan^4 5x}{4} + c = \frac{1}{20} \tan^4 5x + c$$

$$9) \int \sin^4 x \cdot \cos^3 x dx = \int \sin^4 x \cdot (1 - \sin^2 x) \cdot \cos x dx$$

$$= \int \sin^4 x \cdot \cos x dx - \int \sin^6 x \cdot \cos x dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$$

$$\begin{aligned}
 10) \int \frac{\cot^2 \sqrt{x}}{\sqrt{x}} dx &= \int \frac{\csc^2 \sqrt{x} - 1}{\sqrt{x}} dx = 2 \int \frac{\csc^2 \sqrt{x}}{2\sqrt{x}} - \int x^{-1/2} dx \\
 &= 2 \left(-\cot \sqrt{x} \right) - \frac{x^{1/2}}{1/2} + c = -2 \cot \sqrt{x} - 2\sqrt{x} + c
 \end{aligned}$$

5-3- Integrals of inverse trigonometric functions:

The integration formulas for the inverse trigonometric functions are:

$$16) \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + c = -\cos^{-1} \frac{u}{a} + c \quad ; \quad \forall u^2 < a^2$$

$$17) \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + c = -\frac{1}{a} \cot^{-1} \frac{u}{a} + c$$

$$18) \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + c = -\frac{1}{a} \csc^{-1} \left| \frac{u}{a} \right| + c \quad ; \quad \forall u^2 > a^2$$

EX-3 Evaluate the following integrals:

$$1) \int \frac{x^2}{\sqrt{1-x^6}} dx$$

$$6) \int \frac{2dx}{\sqrt{x(1+x)}}$$

$$2) \int \frac{dx}{\sqrt{9-x^2}}$$

$$7) \int \frac{dx}{1+3x^2}$$

$$3) \int \frac{x}{1+x^4} dx$$

$$8) \int \frac{2\cos x}{1+\sin^2 x} dx$$

$$4) \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx$$

$$9) \int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

$$5) \int \frac{dx}{x\sqrt{4x^2-1}}$$

$$10) \int \frac{\tan^{-1} x}{1+x^2} dx$$

Sol.-

$$1) \frac{1}{3} \int \frac{1}{\sqrt{1-(x^3)^2}} (3x^2 dx) = \frac{1}{3} \sin^{-1} x^3 + c$$

$$2) \int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1} \frac{x}{3} + c$$

$$3) \frac{1}{2} \int \frac{2x}{1+(x^2)^2} dx = \frac{1}{2} \tan^{-1} x^2 + c$$

$$4) \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx = \sin^{-1}(\tan x) + c$$

$$5) \int \frac{2 dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}(2x) + c$$

$$6) \int \frac{2}{\sqrt{x}(1+x)} dx = 4 \int \frac{1/2\sqrt{x} dx}{1+(\sqrt{x})^2} = 4 \tan^{-1} \sqrt{x} + c$$

$$7) \frac{1}{\sqrt{3}} \int \frac{\sqrt{3} dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}x) + c$$

$$8) 2 \int \frac{\cos x dx}{1+(\sin x)^2} = 2 \tan^{-1}(\sin x) + c$$

$$9) \int e^{\sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1} x} + c$$

$$10) \int \tan^{-1} x \cdot \frac{dx}{1+x^2} = \frac{(\tan^{-1} x)^2}{2} + c$$

5-4- Integrals of hyperbolic functions:

The integration formulas for the hyperbolic functions are:

$$19) \int \sinh u \cdot du = \cosh u + c$$

$$20) \int \cosh u \cdot du = \sinh u + c$$

$$21) \int \tanh u \cdot du = \ln(\cosh u) + c$$

$$22) \int \coth u \cdot du = \ln(\sinh u) + c$$

$$23) \int \sec^2 u \cdot du = \tanh u + c$$

$$24) \int \csc^2 u \cdot du = \coth u + c$$

$$25) \int \sec hu \cdot \tanh u \cdot du = -\sec hu + c$$

$$26) \int \csc hu \cdot \coth u \cdot du = -\csc hu + c$$

EX-4 – Evaluate the following integrals:

$$1) \int \frac{\cosh(\ln x)}{x} dx$$

$$6) \int \operatorname{sech}^2(2x-3) dx$$

$$2) \int \sinh(2x+1) dx$$

$$7) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$3) \int \frac{\sinh x}{\cosh^4 x} dx$$

$$8) \int (e^{ax} - e^{-ax}) dx$$

$$4) \int x \cdot \cosh(3x^2) dx$$

$$9) \int \frac{\sinh x}{1 + \cosh x} dx$$

$$5) \int \sinh^4 x \cdot \cosh x dx$$

$$10) \int \operatorname{csch}^2 x \cdot \coth x dx$$

Sol.

$$1) \int \cosh(\ln x) \cdot \left(\frac{dx}{x}\right) = \sinh(\ln x) + c$$

$$2) \frac{1}{2} \int \sinh(2x+1) \cdot (2 dx) = \frac{1}{2} \cosh(2x+1) + c$$

$$3) \int \frac{1}{\cosh^3 x} \cdot \frac{\sinh x}{\cosh x} dx = \int \operatorname{sech}^3 x \cdot \tanh x dx$$

$$= - \int \operatorname{sech}^2 x \cdot (-\operatorname{sech} x \cdot \tanh x dx) = -\frac{\operatorname{sech}^3 x}{3} + c$$

$$4) \frac{1}{6} \int \cosh(3x^2) \cdot (6x dx) = \frac{1}{6} \sinh(3x^2) + c$$

$$5) \int \sinh^4 x \cdot (\cosh x dx) = \frac{\sinh^5 x}{5} + c$$

$$6) \frac{1}{2} \int \operatorname{sech}^2(2x-3) \cdot (2 dx) = \frac{1}{2} \tanh(2x-3) + c$$

$$7) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \tanh x dx = \ln(\cosh x) + c$$

$$8) 2 \int \frac{e^{ax} - e^{-ax}}{2} dx = \frac{2}{a} \int \sinh ax (a dx) = \frac{2}{a} \cosh ax + c$$

$$9) \int \frac{\sinh x dx}{1 + \cosh x} = \ln(1 + \cosh x) + c$$

$$10) - \int \operatorname{csch} x \cdot (-\operatorname{csch} x \cdot \coth x dx) = -\frac{\operatorname{csch}^2 x}{2} + c$$

5-5- Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

$$27) \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$28) \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$29) \int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c & \text{if } |u| < 1 \\ \coth^{-1} u + c & \text{if } |u| > 1 \end{cases} = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + c$$

$$30) \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + c = -\cosh^{-1} \left(\frac{1}{|u|} \right) + c$$

$$31) \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + c = -\sinh^{-1} \left(\frac{1}{|u|} \right) + c$$

EX-4 – Evaluate the following integrals:

$$1) \int \frac{dx}{\sqrt{1+4x^2}}$$

$$2) \int \frac{dx}{\sqrt{4+x^2}}$$

$$3) \int \frac{dx}{1-x^2}$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}}$$

$$5) \int \frac{\sec^2 \theta d\theta}{\sqrt{\tan^2 \theta - 1}}$$

$$6) \int \tanh^{-1}(\ln \sqrt{x}) \cdot \frac{dx}{x(1-\ln^2 \sqrt{x})}$$

Sol.-

$$1) \frac{1}{2} \int \frac{2 dx}{\sqrt{1+4x^2}} = \frac{1}{2} \sinh^{-1} 2x + c$$

$$2) \int \frac{\frac{1}{2} dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1} \frac{x}{2} + c$$

$$3) \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \quad \text{if } |x| < 1 \\ = \coth^{-1} x + c \quad \text{if } |x| > 1$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2} \int \frac{\frac{1}{2} dx}{\frac{x}{2}\sqrt{1+\left(\frac{x}{2}\right)^2}} = -\frac{1}{2} \operatorname{cosech}^{-1} \left| \frac{x}{2} \right| + c$$

$$5) \int \frac{1}{\sqrt{\tan^2 \theta - 1}} (\sec^2 \theta d\theta) = \operatorname{cosh}^{-1} (\tan \theta) + c$$

$$6) \text{ let } u = \ln \sqrt{x} = \frac{1}{2} \ln x \quad du = \frac{1}{2x} dx$$

$$\begin{aligned} \int \tanh^{-1} (\ln \sqrt{x}) \cdot \frac{dx}{x(1-\ln^2 \sqrt{x})} &= \int \tanh^{-1} u \cdot \frac{2 du}{1-u^2} \\ &= 2 \frac{(\tanh^{-1} u)^2}{2} + c = \left[\tanh^{-1} (\ln \sqrt{x}) \right]^2 + c \end{aligned}$$

Problems – 5

Evaluate the following integrals:

- | | |
|--|--|
| 1) $\int (x^2 - 1) \cdot (4 - x^2) dx$ | (ans. : $\frac{5}{3}x^3 - \frac{1}{5}x^5 - 4x + c$) |
| 2) $\int e^x \cdot \sin e^x dx$ | (ans. : $-\cos e^x + c$) |
| 3) $\int \tan(3x + 5) dx$ | (ans. : $-\frac{1}{3} \ln \cos(3x + 5) + c$) |
| 4) $\int \frac{\cot(\ln x)}{x} dx$ | (ans. : $\ln \sin(\ln x) + c$) |
| 5) $\int \frac{\sin x + \cos x}{\cos x} dx$ | (ans. : $-\ln \cos x + x + c$) |
| 6) $\int \frac{dx}{1 + \cos x}$ | (ans. : $-\cot x + \csc x + c$) |
| 7) $\int \cot(2x + 1) \cdot \csc^2(2x + 1) dx$ | (ans. : $-\frac{1}{4} \cot^2(2x + 1) + c$) |
| 8) $\int \frac{dx}{\sqrt{1 - 9x^2}}$ | (ans. : $\frac{1}{3} \sin^{-1}(3x) + c$) |
| 9) $\int \frac{dx}{\sqrt{2 - x^2}}$ | (ans. : $\sin^{-1} \frac{x}{\sqrt{2}} + c$) |
| 10) $\int e^{2x} \cdot \cos e^{2x} dx$ | (ans. : $\frac{1}{2} \sinh e^{2x} + c$) |
| 11) $\int e^{\sin x} \cdot \cos x dx$ | (ans. : $e^{\sin x} + c$) |
| 12) $\int \frac{dx}{e^{3x}}$ | (ans. : $-\frac{1}{3} e^{-3x} + c$) |
| 13) $\int \frac{e^{\sqrt{x}} - 1}{\sqrt{x}} dx$ | (ans. : $2e^{\sqrt{x}} - 2\sqrt{x} + c$) |
| 14) $\int x(a + b\sqrt{3x}) dx$ where a, b constants | (ans. : $\frac{1}{10} (5ax^2 + 4\sqrt{3}bx^{5/2}) + c$) |
| 15) $\int \frac{dx}{-1 - x^2}$ | (ans. : $-\tan^{-1} x + c$) |
| 16) $\int \frac{\cos \theta d\theta}{1 + \sin^2 \theta}$ | (ans. : $\tan^{-1}(\sin \theta) + c$) |

- 17) $\int \frac{1}{x^2} \csc \frac{1}{x} \cot \frac{1}{x} dx$ (ans.: $\csc \frac{1}{x} + c$)
- 18) $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}} dx$ (ans.: $\frac{3}{4} \sqrt[3]{(3x^2+2x+1)^2} + c$)
- 19) $\int \sin(\tan \theta) \cdot \sec^2 \theta d\theta$ (ans.: $-\cos(\tan \theta) + c$)
- 20) $\int \sqrt{x^2 - x^4} dx$ (ans.: $-\frac{1}{3} \sqrt{(1-x^2)^3} + c$)
- 21) $\int \frac{\sec^2 2x dx}{\sqrt{\tan 2x}}$ (ans.: $\sqrt{\tan 2x} + c$)
- 22) $\int (\sin \theta - \cos \theta)^2 d\theta$ (ans.: $\theta + \cos^2 \theta + c$)
- 23) $\int \frac{y}{y^4 + 1} dy$ (ans.: $\frac{1}{2} \tan^{-1} y^2 + c$)
- 24) $\int \frac{dx}{\sqrt{x(x+1)}}$ (ans.: $2 \tan^{-1} \sqrt{x} + c$)
- 25) $\int t^{\frac{2}{3}} (t^{\frac{5}{3}} + 1)^{\frac{2}{3}} dt$ (ans.: $\frac{9}{25} (t^{\frac{5}{3}} + 1)^{\frac{5}{3}} + c$)
- 26) $\int \frac{dx}{x^{\frac{1}{5}} \sqrt{1+x^{\frac{4}{5}}}}$ (ans.: $\frac{5}{2} \sqrt{1+x^{\frac{4}{5}}} + c$)
- 27) $\int \frac{(\cos^{-1} 4x)^2}{\sqrt{1-16x^2}} dx$ (ans.: $-\frac{1}{12} (\cos^{-1} 4x)^3 + c$)
- 28) $\int \frac{dx}{x\sqrt{4x^2-1}}$ (ans.: $\sec^{-1}(2x) + c$)
- 29) $\int \frac{dx}{(e^x + e^{-x})^2}$ (ans.: $\frac{1}{4} \tanh x + c$)
- 30) $\int 3^{\ln x^2} \frac{dx}{x}$ (ans.: $\frac{1}{2 \ln 3} 3^{\ln x^2} + c$)
- 31) $\int \frac{\cot x dx}{\ln(\sin x)}$ (ans.: $\ln \ln(\sin x) + c$)
- 32) $\int \frac{(\ln x)^2}{x} dx$ (ans.: $\frac{1}{3} (\ln x)^3 + c$)
- 33) $\int \frac{\sin x \cdot e^{\sec x}}{\cos^2 x} dx$ (ans.: $e^{\sec x} + c$)

- 34) $\int \frac{dx}{x \cdot \ln x}$ (ans. : $\ln \ln x + c$)
- 35) $\int \frac{d\theta}{\cosh \theta + \sinh \theta}$ (ans. : $-e^{-\theta} + c$)
- 36) $\int \frac{2^x - 8^{2x}}{\sqrt{4^x}} dx$ (ans. : $x - \frac{1}{5 \ln 2} 2^{5x} + c$)
- 37) $\int \frac{e^{\tan^{-1} 2t}}{1 + 4t^2} dt$ (ans. : $\frac{1}{2} e^{\tan^{-1} 2t} + c$)
- 38) $\int \frac{\cot x}{\csc x} dx$ (ans. : $\sin x + c$)
- 39) $\int \sec^4 x \cdot \tan^3 x dx$ (ans. : $\frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + c$)
- 40) $\int \csc^4 3x dx$ (ans. : $-\frac{1}{9} \cot^3 3x - \frac{1}{3} \cot 3x + c$)
- 41) $\int \frac{\cos^3 t}{\sin^2 t} dt$ (ans. : $-\operatorname{csc} t - \sin t + c$)
- 42) $\int \frac{\sec^4 x}{\tan^4 x} dx$ (ans. : $-\frac{1}{3} \cot^3 x - \cot x + c$)
- 43) $\int \tan^2 4\theta d\theta$ (ans. : $\frac{1}{4} \tan 4\theta - \theta + c$)
- 44) $\int \frac{e^x}{1 + e^x} dx$ (ans. : $\ln(1 + e^x) + c$)
- 45) $\int \tan^3 2x dx$ (ans. : $\frac{1}{4} \tan^2 2x + \frac{1}{2} \ln|\cos 2x| + c$)
- 46) $\int \frac{\sec^2 x}{2 + \tan x} dx$ (ans. : $\ln(2 + \tan x) + c$)
- 47) $\int \sec^4 3x dx$ (ans. : $\frac{1}{9} \tan^3 3x + \frac{1}{3} \tan 3x + c$)
- 48) $\int \frac{e^t}{1 + e^{2t}} dt$ (ans. : $\tan^{-1} e^t + c$)
- 49) $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ (ans. : $2 \sin \sqrt{x} + c$)
- 50) $\int \frac{dx}{\sin x \cdot \cos x}$ (ans. : $-\ln|\csc 2x + \cot 2x| + c$)

$$51) \int \sqrt{1 + \sin y} \, dy \quad (\text{ans. : } -2\sqrt{1 - \sin y} + c)$$

$$52) \int \frac{dx}{(x^2 + 1)(2 + \tan^{-1} x)} \quad (\text{ans. : } \ln(2 + \tan^{-1} x) + c)$$

$$53) \int \sin^{-1}(\cosh x) \cdot \frac{\sinh x \, dx}{\sqrt{1 - \cosh^2 x}} \quad (\text{ans. : } \frac{1}{2} (\sinh^{-1}(\cosh x))^2 + c)$$

$$54) \int \frac{\cos \theta \, d\theta}{1 - \sin^2 \theta} \quad (\text{ans. : } \ln|\sec \theta + \tan \theta| + c)$$

$$55) \int \frac{dx}{x(1 + (\ln x)^2)} \quad (\text{ans. : } \tan^{-1}(\ln x) + c)$$

$$56) \int \left(e^{\frac{2}{4}x} - 2e^{\frac{5}{4}x} + e^{\frac{x}{4}} \right) dx \quad (\text{ans. : } \frac{4}{9}e^{\frac{2}{4}x} - \frac{8}{5}e^{\frac{5}{4}x} + 4e^{\frac{x}{4}} + c)$$

$$57) \int \frac{e^x \, dx}{e^{2x} + 2e^x + 1} \quad (\text{ans. : } -\frac{1}{e^x + 1} + c)$$

$$58) \int e^x \cdot \sinh 2x \, dx \quad (\text{ans. : } \frac{1}{2} \left[\frac{1}{3}e^{3x} + e^{-x} \right] + c)$$

$$59) \int \frac{\sec^3 x + e^{\sin x}}{\sec x} \, dx \quad (\text{ans. : } \tan x + e^{\sin x} + c)$$

$$60) \int \frac{3^{x+2}}{2 + 9^{x+1}} \, dx \quad (\text{ans. : } \frac{3}{\sqrt{2} \ln 3} \tan^{-1} \frac{3^{x+1}}{\sqrt{2}} + c)$$

$$61) \int \frac{\cos x \, dx}{\sqrt{\sin x} \cdot \sqrt{1 - \sin x}} \quad (\text{ans. : } 2\sin^{-1} \sqrt{\sin x} + c)$$

$$62) \int \tan^5 x \, dx \quad (\text{ans. : } \frac{1}{4} \sec^4 x - \sec^2 x - \ln|\cos x| + c)$$

$$63) \int e^{\ln \sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}} \quad (\text{ans. : } \frac{1}{2} (\sin^{-1} x)^2 + c)$$

$$64) \int x \cdot e^{x^2-1} \, dx \quad (\text{ans. : } \frac{1}{2} e^{x^2-1} + c)$$

$$65) \int \cosh(\ln \cos x) \, dx \quad (\text{ans. : } \frac{1}{2} [\sin x + \ln|\sec x + \tan x|] + c)$$

$$66) \int \frac{\cos x}{\sin^2 x} \, dx \quad (\text{ans. : } -\csc x + c)$$

$$67) \int \cosh^{-1}(\sin x) \frac{\cos x \, dx}{\sqrt{\sin^2 x - 1}} \quad (\text{ans. : } \frac{1}{2} [\cosh^{-1}(\sin x)]^2 + c)$$