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ENGINEERING ANALYSIS
THIRD STAGE
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LECTURE 1

LAPLACE TRANSFORM-1

The Laplace transform of an expression $f(t)$ is denoted by $L\{f(t)\}$ and is defined as the semi-infinite integral

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt$$

In determining the transform of an expression, you will appreciate that the limits of the integral are substituted for t, so that the result will be an expression in s. Therefore

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt = F(s)$$

So we have $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$

Example 1

To find the Laplace transform of $f(t) = a$ (constant).

$$L\{a\} = \int_0^\infty ae^{-st}\,dt = a\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{a}{s}\left[e^{-st}\right]_0^\infty$$

$$= -\frac{a}{s}\{0-1\} = \frac{a}{s}$$

$$\therefore\ L\{a\} = \frac{a}{s} \qquad (s > 0)$$

Example 2

To find the Laplace transform of $f(t) = e^{at}$ (a constant). As with all cases, we multiply $f(t)$ by e^{-st} and integrate between $t = 0$ and $t = \infty$.

$$\therefore\ L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= -\frac{1}{s-a}\{0-1\} = \frac{1}{s-a}$$

$$L\{e^{at}\} = \frac{1}{s-a} \qquad (s > a)$$

So, according to example 1 and two we can write the following:

$$L\{4\} = \frac{4}{s}; \qquad L\{e^{4t}\} = \frac{1}{s-4}$$
$$L\{-5\} = -\frac{5}{s}; \qquad L\{e^{-2t}\} = \frac{1}{s+2}$$

Note that, as we said earlier, the Laplace transform is always an expression in s.

Example 3

To find the Laplace transform of $f(t) = \sin at$. We could, of course, apply the definition and evaluate

$$L\{\sin at\} = \int_0^\infty \sin at \cdot e^{-st}\,dt$$

using integration by parts.

However, it is much shorter if we use the fact that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

so that $\sin\theta$ is the imaginary part of $e^{j\theta}$, written $\mathscr{I}(e^{j\theta})$.

The function $\sin at$ can therefore be written $\mathscr{I}(e^{jat})$ so that

$$L\{\sin at\} = L\{\mathscr{I}(e^{jat})\} = \mathscr{I}\int_0^\infty e^{jat}e^{-st}\,dt = \mathscr{I}\int_0^\infty e^{-(s-ja)t}\,dt$$
$$= \mathscr{I}\left\{\left[\frac{e^{-(s-ja)t}}{-(s-ja)}\right]_0^\infty\right\} = \mathscr{I}\left\{-\frac{1}{(s-ja)}[0-1]\right\}$$
$$= \mathscr{I}\left\{\frac{1}{s-ja}\right\}$$

We can rationalise the denominator by multiplying top and bottom by . . . .

$$\boxed{s + ja}$$

$$\therefore\ L\{\sin at\} = \mathscr{I}\left\{\frac{s + ja}{s^2 + a^2}\right\} = \frac{a}{s^2 + a^2}$$ **(only the imaginary part)**

$$\therefore\ L\{\sin at\} = \frac{a}{s^2 + a^2}$$

As the (sin at) is the imaginary part so we select the imaginary term from(s + ja) which is equal to (a). and we can solve this by integration by part. We can use the same method to determine

L(cos at) as it the real part so,

$$L\{\cos at\} = \Re\left\{\frac{s + ja}{s^2 + a^2}\right\} = \frac{s}{s^2 + a^2}$$

Accordingly the Laplace transform of the following are as follows:

$$\boxed{\begin{aligned} L\{1\} &= \frac{1}{s}; & L\{e^{3t}\} &= \frac{1}{s-3} \\ L\{\sin 2t\} &= \frac{2}{s^2+4}; & L\{\cos 4t\} &= \frac{s}{s^2+16} \end{aligned}}$$

Example 4

To find the transform of $f(t) = t^n$ where n is a positive integer.

By the definition $L\{t^n\} = \int_0^\infty t^n e^{-st}\,dt.$

Integrating by parts

$$L\{t^n\} = \left[t^n\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt$$

$$= -\frac{1}{s}\left[t^n e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$$

We said earlier that in a product such as $t^n e^{-st}$ the numerical value of s is large enough to make the product converge to zero as $t \to \infty$

$$\therefore \left[t^n e^{-st}\right]_0^\infty = 0 - 0 = 0$$

$$\therefore L\{t^n\} = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$$

You will notice that $\int_0^\infty t^{n-1}e^{-st}\,dt$ is identical to $\int_0^\infty t^n e^{-st}\,dt$ except that n is replaced by $(n-1)$.

$\therefore$ If $I_n = \int_0^\infty t^n e^{-st}\,dt$, then $I_{n-1} = \int_0^\infty t^{n-1}e^{-st}\,dt$

So $I_n = n/s * I_{n-1}$ this is from the following formula:

$$\therefore L\{t^n\} = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$$

S0,

$I_{n=}$ n/s * I_{n-1}

$I_{n-1=}$ n-1/s *I_{n-2}

$I_{n-2=}$ n-2/s *I_{n-3} etc…. then

$$\text{So } I_n = \int_0^\infty t^n e^{-st}\,dt = \frac{n}{s}.I_{n-1}$$
$$= \frac{n}{s}.\frac{n-1}{s}.I_{n-2}$$
$$= \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.I_{n-3} \quad \text{etc.}$$

$$\boxed{I_n = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s}.I_{n-4}}$$

So finally, we have

$$I_n = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s}\ldots\frac{n-(n-1)}{s}.I_0$$

But $I_0 = L\{t^0\} = L\{1\} = \frac{1}{s}$

$$\therefore\ I_n = \frac{n(n-1)(n-2)(n-3)\ldots(3)(2)(1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$\therefore\ L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\therefore\ L\{t\} = \frac{1}{s^2};\quad L\{t^2\} = \frac{2}{s^3};\quad L\{t^3\} = \frac{6}{s^4}$$

and with $n = 0$, since $0! = 1$, the general result includes $L\{1\} = \frac{1}{s}$ which we have already established.

Example 5

Laplace transforms of $f(t) = \sinh at$ and $f(t) = \cosh at$.

Starting from the exponential definitions of $\sinh at$ and $\cosh at$, i.e.

$$\sinh at = \tfrac{1}{2}(e^{at} - e^{-at}) \quad \text{and} \quad \cosh at = \tfrac{1}{2}(e^{at} + e^{-at})$$

$$\therefore \; L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

$$L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

So, we summarized the previous by the followings:

$$L\{a\} = \frac{a}{s}; \qquad L\{e^{at}\} = \frac{1}{s-a}; \qquad L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{\sin at\} = \frac{a}{s^2 + a^2}; \qquad L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}; \qquad L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is*

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is*

$$L\{kf(t)\} = kL\{f(t)\}$$

Example 6

(a) $$\begin{aligned} L\{2e^{-t} + t\} &= L\{2e^{-t}\} + L\{t\} \\ &= 2L\{e^{-t}\} + L\{t\} \\ &= \frac{2}{s+1} + \frac{1}{s^2} = \frac{2s^2+s+1}{s^2(s+1)} \end{aligned}$$

(b) $$\begin{aligned} L\{2\sin 3t + \cos 3t\} &= 2L\{\sin 3t\} + L\{\cos 3t\} \\ &= 2.\frac{3}{s^2+9} + \frac{s}{s^2+9} = \frac{s+6}{s^2+9} \end{aligned}$$

(c) $$\begin{aligned} L\{4e^{2t} + 3\cosh 4t\} &= 4L\{e^{2t}\} + 3L\{\cosh 4t\} \\ &= 4.\frac{1}{s-2} + 3.\frac{s}{s^2-16} = \frac{4}{s-2} + \frac{3s}{s^2-16} \\ &= \frac{7s^2-6s-64}{(s-2)(s^2-16)} \end{aligned}$$

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1. $L\{2\sin 3t + 4\sinh 3t\} = 2.\frac{3}{s^2+9} + 4.\frac{3}{s^2-9}$

$$= \frac{6}{s^2+9} + \frac{12}{s^2-9} = \frac{18(s^2+3)}{s^4-81}$$

2. $L\{5e^{4t} + \cosh 2t\} = \frac{5}{s-4} + \frac{s}{s^2-4} = \frac{6s^2-4s-20}{(s-4)(s^2-4)}$

3. $L\{t^3 + 2t^2 - 4t + 1\} = \frac{3!}{s^4} + 2.\frac{2!}{s^3} - 4.\frac{1!}{s^2} + \frac{1}{s}$

$$= \frac{1}{s^4}\{s^3 - 4s^2 + 4s + 6\}$$

Theorem 1 The first shift theorem

The first shift theorem states that if $L\{f(t)\} = F(s)$ then
$L\{e^{-at}f(t)\} = F(s+a)$

Because $L\{e^{-at}f(t)\} = \int_{t=0}^{\infty} e^{-at}f(t)e^{-st}\,dt = \int_{t=0}^{\infty} f(t)e^{-(s+a)t}\,dt = F(s+a)$

That is

$L\{e^{-at}f(t)\} = F(s+a)$

The transform $L\{e^{-at}f(t)\}$ is thus the same as $L\{f(t)\}$ with s everywhere in the result replaced by $(s+a)$.

For example $L\{\sin 2t\} = \frac{2}{s^2+4}$

then $L\{e^{-3t}\sin 2t\} = \frac{2}{(s+3)^2+4} = \frac{2}{s^2+6s+13}$

Because $L\{t^2\} = \frac{2}{s^3}$. $\therefore$ $L\{t^2e^{4t}\}$ is the same with s replaced by $(s-4)$.

$$\therefore L\{t^2e^{4t}\} = \frac{2}{(s-4)^3}$$

EXAMPLES:

1. $L\{\cosh 3t\} = \frac{s}{s^2-9}$ $\quad \therefore L\{e^{-2t}\cosh 3t\} = \frac{s+2}{(s+2)^2-9} = \frac{s+2}{s^2+4s-5}$

2. $L\{\sin 3t\} = \frac{3}{s^2+9}$ $\quad \therefore L\{2e^{3t}\sin 3t\} = \frac{6}{(s-3)^2+9} = \frac{6}{s^2-6s+18}$

3. $L\{4t\} = 4.\frac{1}{s^2}$ $\quad \therefore L\{4te^{-t}\} = \frac{4}{(s+1)^2}$

4. $L\{\cos t\} = \frac{s}{s^2+1}$ $\quad \therefore L\{e^{2t}\cos t\} = \frac{s-2}{(s-2)^2+1} = \frac{s-2}{s^2-4s+5}$

5. $L\{\sinh 2t\} = \frac{2}{s^2-4}$ $\quad \therefore L\{e^{3t}\sinh 2t\} = \frac{2}{(s-3)^2-4} = \frac{2}{s^2-6s+5}$

6. $L\{t^3\} = \frac{3!}{s^4}$ $\quad \therefore L\{t^3e^{-4t}\} = \frac{6}{(s+4)^4}$

Theorem 2 Multiplying by *t* and t^n

If $L\{f(t)\} = F(s)$ then $L\{tf(t)\} = -F'(s)$

$$\text{Because } L\{tf(t)\} = \int_{t=0}^{\infty} tf(t)e^{-st}\,dt = \int_{t=0}^{\infty} f(t)\left(-\frac{de^{-st}}{ds}\right)dt$$

$$= -\frac{d}{ds}\int_{t=0}^{\infty} f(t)e^{-st}\,dt = -F'(s)$$

That is

$$L\{tf(t)\} = -F'(s)$$

For example, $L\{\sin 2t\} = \dfrac{2}{s^2+4}$

$$\therefore\ L\{t\sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

Theorem 3 Dividing by *t*

If $L\{f(t)\} = F(s)$ then $L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma$

provided $\displaystyle\lim_{t\to 0}\left(\frac{f(t)}{t}\right)$ exists. To demonstrate this we start from the right-hand side of the result

$$\int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma = \int_{\sigma=s}^{\infty}\left\{\int_{t=0}^{\infty} f(t)e^{-\sigma t}\,dt\right\}d\sigma$$

$$= \int_{t=0}^{\infty}\int_{\sigma=s}^{\infty} f(t)e^{-\sigma t}\,d\sigma\,dt$$

$$= \int_{t=0}^{\infty} f(t)\left\{\int_{\sigma=s}^{\infty} e^{-\sigma t}\,d\sigma\right\}dt$$

$$= \int_{t=0}^{\infty} f(t)\frac{e^{-st}}{t}\,dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

Notice the dummy variable σ. The end result is an expression in s which comes from the lower limit of the integral so the variable of integration, which is absorbed during the process of integration, is changed to σ. Notice also that we interchange the order of integration.

This rule is somewhat restricted in use, since it is applicable only if $\displaystyle\lim_{t\to 0}\left(\frac{f(t)}{t}\right)$ exists. In indeterminate cases, we use L'Hôpital's rule to find out. Let's try a couple of examples.

Example 1

Determine $L\left\{\dfrac{\sin at}{t}\right\}$

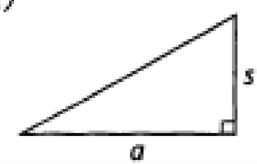
First we test $\underset{t\to 0}{Lim}\left\{\dfrac{\sin at}{t}\right\} = \left\{\dfrac{0}{0}\right\} = ?$ By L'Hôpital's rule, we differentiate top and bottom separately and substitute $t = 0$ in the result to ascertain the limit of the new expression.

$\underset{t\to 0}{Lim}\left\{\dfrac{\sin at}{t}\right\} = \underset{t\to 0}{Lim}\left\{\dfrac{a\cos at}{1}\right\} = a$, that is, the limit exists and the theorem can therefore be applied.

So $L\{\sin at\} = \dfrac{a}{s^2 + a^2}$, therefore

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{\sigma^2 + a^2}\,d\sigma = \left[\arctan\left(\frac{\sigma}{a}\right)\right]_s^\infty = \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right) = \arctan\left(\frac{a}{s}\right)$$

Notice that $\arctan\left(\dfrac{a}{s}\right) + \arctan\left(\dfrac{s}{a}\right) = \dfrac{\pi}{2}$, as can be seen from the figure



Example 2

Determine $L\left\{\dfrac{1 - \cos 2t}{t}\right\}$

First we test whether $\underset{t\to 0}{Lim}\left\{\dfrac{1 - \cos 2t}{t}\right\}$ exists. Result? .

$$\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\} = \frac{1-1}{0} = \frac{0}{0} = ? \quad \therefore \text{ Apply l'Hôpital's rule.}$$

$$\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\} = \lim_{t\to 0}\left\{\frac{2\sin 2t}{1}\right\} = \frac{0}{1} = 0 \quad \therefore \text{ limit exists.}$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

Then, by Theorem 3

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_{\sigma=s}^{\infty}\left\{\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+4}\right\}\mathrm{d}\sigma$$

$$= \left[\ln\sigma - \frac{1}{2}\ln(\sigma^2+4)\right]_{\sigma=s}^{\infty} = \frac{1}{2}\left[\ln\left(\frac{\sigma^2}{\sigma^2+4}\right)\right]_{\sigma=s}^{\infty}$$

When $\sigma \to \infty$, $\ln\left(\frac{\sigma^2}{\sigma^2+4}\right) \to \ln 1 = 0$

Therefore, $L\left\{\frac{1-\cos 2t}{t}\right\} = \ldots\ldots\ldots\ldots$

Because

$$L\left\{\frac{1-\cos 2t}{t}\right\} = -\frac{1}{2}\ln\left(\frac{s^2}{s^2+4}\right) = \ln\left(\frac{s^2}{s^2+4}\right)^{-1/2}$$

$$= \ln\sqrt{\frac{s^2+4}{s^2}}$$

1 *Standard transforms*

$f(t)$	$L\{f(t)\} = F(s)$	
a	$\dfrac{a}{s}$	
e^{at}	$\dfrac{1}{s-a}$	
$\sin at$	$\dfrac{a}{s^2+a^2}$	
$\cos at$	$\dfrac{s}{s^2+a^2}$	
$\sinh at$	$\dfrac{a}{s^2-a^2}$	
$\cosh at$	$\dfrac{s}{s^2-a^2}$	
t^n	$\dfrac{n!}{s^{n+1}}$	(n a positive integer)

2 *Theorem 1* The first shift theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at}f(t)\} = F(s+a)$

3 *Theorem 2* Multiplying by t

If $L\{f(t)\} = F(s)$, then $L\{tf(t)\} = -\dfrac{d}{ds}\{F(s)\}$

4 *Theorem 3* Dividing by t

If $L\{f(t)\} = F(s)$, then $L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma$

provided $\displaystyle\lim_{t\to 0}\left\{\dfrac{f(t)}{t}\right\}$ exists.

Exercise

Determine the Laplace transforms of the following expressions.

1 $\sin 3t$

2 $\cos 2t$

3 e^{At}

4 $6t^2$

5 $\sinh 3t$

6 $t\cosh 4t$

7 $t^2 - 3t + 4$

8 $\dfrac{e^{3t} - 1}{t}$

9 $e^{3t}\cos 4t$

10 $t^2 \sin t$

Results:

1 $\dfrac{3}{s^2+9}$

2 $\dfrac{s}{s^2+4}$

3 $\dfrac{1}{s-4}$

4 $\dfrac{12}{s^3}$

5 $\dfrac{3}{s^2-9}$

6 $\dfrac{s^2+16}{(s^2-16)^2}$

7 $\dfrac{1}{s^3}(4s^2-3s+2)$

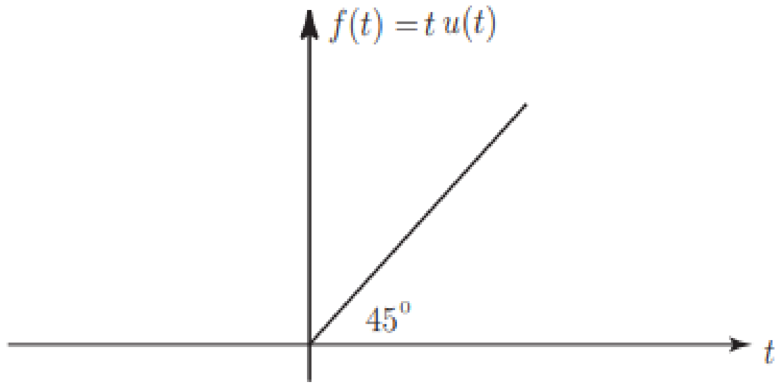
8 $\ln\left(\dfrac{s}{s-3}\right)$

9 $\dfrac{s-3}{s^2-6s+25}$

10 $\dfrac{6s^2-2}{(s^2+1)^3}$

LAPLACE TRANSFORM OF RAMP FUNCTION:

To begin, we determine the Laplace transform of some simple causal functions. For example, if we consider the **ramp function** $f(t) = t.u(t)$ with graph



we find:

$$\begin{aligned}
\mathcal{L}\{t\ u(t)\} &= \int_0^\infty \mathrm{e}^{-st} t\ u(t)\,dt \\
&= \int_0^\infty \mathrm{e}^{-st} t\ dt \qquad \text{since in the range of the integral } u(t) = 1 \\
&= \left[\frac{t\ \mathrm{e}^{-st}}{(-s)}\right]_0^\infty - \int_0^\infty \frac{\mathrm{e}^{-st}}{(-s)}\,dt \qquad \text{using integration by parts} \\
&= \left[\frac{t\mathrm{e}^{-st}}{(-s)}\right]_0^\infty - \left[\frac{\mathrm{e}^{-st}}{(-s)^2}\right]_0^\infty
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{t\ u(t)\} &= [0-0] - \left[0 - \left(\frac{1}{(-s)^2}\right)\right] \\
&= \frac{1}{s^2}
\end{aligned}$$

Thus, if f(t) = t u(t) then F(s) = 1/s2 .

A similar, but more tedious, calculation yields the result that if $f(t) = t^n u(t)$ in which n is a positive integer then:

$L\{t^n u(t)\} = n!/ s^{n+1}$

[We remember n! ≡ n(n − 1)(n − 2). . .(3)(2)(1).]

EXAMPLE: Find the Laplace transform of the step function u(t).

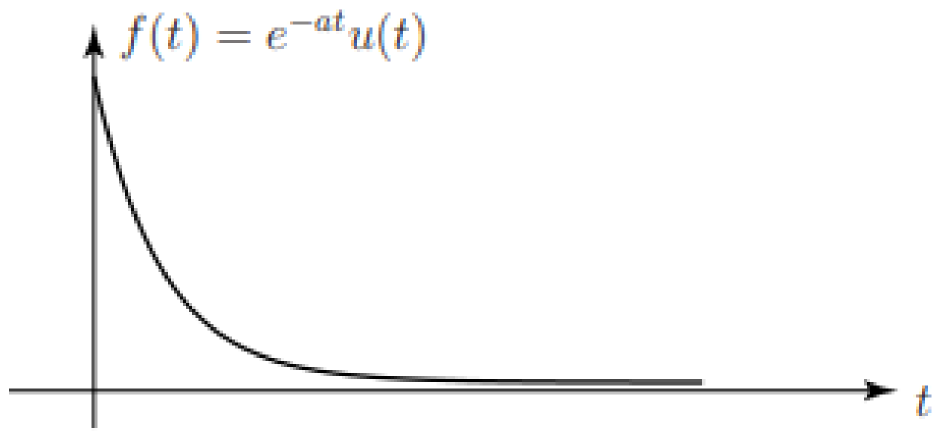
You should obtain $\int_0^\infty e^{-st}\, dt$ since in the range of integration, $t > 0$ and so $u(t) = 1$ leading to

$$\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} u(t)\ dt = \int_0^\infty e^{-st}\, dt$$

$$\begin{aligned}\mathcal{L}\{u(t)\} &= \int_0^\infty e^{-st}\, dt \\ &= \left[\frac{e^{-st}}{(-s)}\right]_0^\infty = 0 - \left[\frac{1}{(-s)}\right] = \frac{1}{s}\end{aligned}$$

EXAMPLE: Find the Laplace transform of the following casual function, means the function $f(t)= e^{-at}$ is well defined in the negative region but if we multiply it by the unit step function it will convert to be defined in the positive region only as the function and its graph.

f(t)= e^{-at} u(t)



In this case,

$$\begin{aligned}
\mathcal{L}\{e^{-at}u(t)\} &= \int_0^\infty e^{-st}e^{-at}\,dt \\
&= \int_0^\infty e^{-(s+a)t}\,dt \\
&= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a} \qquad \text{(zero contribution from the upper limit)}
\end{aligned}$$

Therefore, if $f(t) = e^{-at}u(t)$ then $F(s) = \dfrac{1}{s+a}$.

The linearity property of the Laplace transformation

If $f(t)$ and $g(t)$ are causal functions and c_1, c_2 are constants then

$$\begin{aligned}
\mathcal{L}\{c_1f(t) + c_2g(t)\} &= \int_0^\infty e^{-st}[c_1f(t) + c_2g(t)]\,dt \\
&= c_1\int_0^\infty e^{-st}f(t)\,dt + c_2\int_0^\infty e^{-st}g(t)\,dt \\
&= c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}
\end{aligned}$$

TABLE OF SOME CASUAL FUNCTIONS LAPLACE TRANSFORM

Rule	Causal function	Laplace transform
1	$f(t)$	$F(s)$
2	$u(t)$	$\frac{1}{s}$
3	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
4	$e^{-at} u(t)$	$\frac{1}{s+a}$
5	$\sin at \,.\, u(t)$	$\frac{a}{s^2+a^2}$
6	$\cos at \,.\, u(t)$	$\frac{s}{s^2+a^2}$
7	$e^{-at} \sin bt \,.\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
8	$e^{-at} \cos bt \, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$

EXAMPLE: Find the Laplace transform of the following casual function:

$$\begin{aligned}\mathcal{L}\{2\cos t \,.\, u(t) - 3t^2u(t)\} &= 2\mathcal{L}\{\cos t \,.\, u(t)\} - 3\mathcal{L}\{t^2u(t)\} \\ &= 2\left(\frac{s}{s^2+1}\right) - 3\left(\frac{2}{s^3}\right)\end{aligned}$$

EXAMPLE: Find the Laplace transform of the delayed step-function u(t − a), a > 0.

Answer

You should obtain $\mathcal{L}\{u(t-a)\} = \int_a^\infty e^{-st}\,dt$ (note the lower limit is a) since:

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_0^a e^{-st}u(t-a)\,dt + \int_a^\infty e^{-st}u(t-a)\,dt$$

In the first integral $0 < t < a$ and so $(t-a) < 0$, therefore $u(t-a) = 0$.

In the second integral $a < t < \infty$ and so $(t-a) > 0$, therefore $u(t-a) = 1$. Hence

$$\mathcal{L}\{u(t-a)\} = 0 + \int_a^\infty e^{-st}\,dt.$$

$$= \int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{(-s)}\right]_a^\infty = \frac{e^{-sa}}{s}$$

END