Fourier analysis

Frequency Domain Representation

1. Frequency response to a complex exponential sequence:

Let the input to the linear shift invariant system be the complex exponential sequence; $x(n) = e^{jwn}$. the corresponding output of LSI system with impulse response $h(n)$ is given by convolution sum as follows:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \quad \& \quad x(n) = e^{j\omega n}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega n}e^{-j\omega k}$$

$$y(n) = e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$y(n) = x(n) * H(e^{j\omega})$$

And the frequency Response is

$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

- $H\left(e^{j\omega}\right)$ is called frequency response of linear shift invariant system.
- The output can be written in terms of $H\left(e^{j\omega}\right)$ as

$$y(n) = e^{j\omega n}.H(e^{j\omega})$$

- The output $y(n)$ is a product of input signal $x(n)$ and frequency response $H(e^{j\omega})$.
- In general, $H(e^{j\omega})$ is a complex variable for each w and can be given either rectangular and exponential form.

$$H(e^{j\omega}) = H_{Re}(e^{j\omega}) + JH_{Im}(e^{j\omega})$$

$$= |H(e^{j\omega})| \exp\,[j\ argH(e^{j\omega})]$$

Where the magnitude and phase of $H(e^{j\omega})$ are

magnitude $\longrightarrow$ $|H(e^{j\omega})| = \sqrt{H_R^2(e^{j\omega}) + H_{Im}^2(e^{j\omega})}$

phase $\longrightarrow$ $argH(e^{j\omega}) = tan^{-1}[\frac{H_{Im}(e^{j\omega})}{H_{Re}(e^{j\omega})}]$

Encountered Series

$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$	$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad \lvert a\rvert < 1$
$\sum_{n=0}^{N-1} na^n = \frac{(N-1)a^{N+1} - Na^N + a}{(1-a)^2}$	$\sum_{n=0}^{\infty} na^n = \frac{a}{(1-a)^2} \qquad \lvert a\rvert < 1$
$\sum_{n=0}^{N-1} n = \frac{1}{2}N(N-1)$	$\sum_{n=0}^{N-1} n^2 = \frac{1}{6}N(N-1)(2N-1)$

<u>EX:</u> Determine the frequency response of $h(n) = u(n)$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} u(n)e^{-j\omega n}$$

$$= \sum_{k=0}^{\infty} 1.\, e^{-j\omega n}$$

$$H\left(e^{j\omega}\right) = \frac{1}{1 - e^{-j\omega}}$$

Ex2 Consider the LSI system 1- Find the DTFT

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n+3)$$

Sol:-

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jnw}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u(n+3)\, e^{-jnw}$$

$$X(e^{jw}) = \sum_{n=-3}^{\infty} \left(\tfrac{1}{2}\right)^n e^{-jnw}$$

$$X(e^{jw}) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{n-3} e^{-j(n-3)w}$$

$$X(e^{jw}) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \cdot \left(\tfrac{1}{2}\right)^{-3} \cdot e^{-jnw} \cdot e^{3jw}$$

$$= \left(\tfrac{1}{2} e^{jw}\right)^{-3} \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \cdot e^{-jnw}$$

$$= \left(\tfrac{1}{2} e^{jw}\right)^{-3} \sum_{n=0}^{\infty} \left(\tfrac{1}{2} e^{-jw}\right)^n$$

$$\boxed{= \left(\tfrac{1}{2} e^{jw}\right)^{-3} \cdot \frac{1}{1 - \frac{1}{2} e^{-jw}}}$$

EX: Determine the frequency response of the following system

$$h(n) = \delta(n) + \delta(n-2)$$

Solution:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} [\delta(n) + \delta(n-2)]e^{-j\omega n}$$

$$H(e^{j\omega}) = e^{0} + 1.e^{-2j\omega} = 1 + e^{-2j\omega}$$

$$H(e^{j\omega}) = 1 + cos2\omega + jsin2\omega$$

Some Geometric Series

$$\sum_{n=-\infty}^{\infty} na^{n}e^{-j\omega n} = \frac{a}{(1-a)^2}$$

$$\sum_{n=-\infty}^{\infty} n^{2}e^{-j\omega n} = \frac{1}{6}n(n-1)(2n-1)$$