



Class: 2nd

Subject: Strength of Materials

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Strength of Materials

Second Stage

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Introduction

- ▶ In this chapter we will concentrate on the behavior of slender members subjected to torsional loading that is, loading by couples that produce twisting of the member about its axis.
- ▶ Figure (4.1) shows a common example of torsional loading and indicates the shear stresses and the stress resultant associated with torsion.
- ▶ In Figure (4.1), shaft AB is say a *torsion member*.
- ▶ The shaft AB is subjected to equal and opposite torques of magnitude T that twist one end relative to the other.

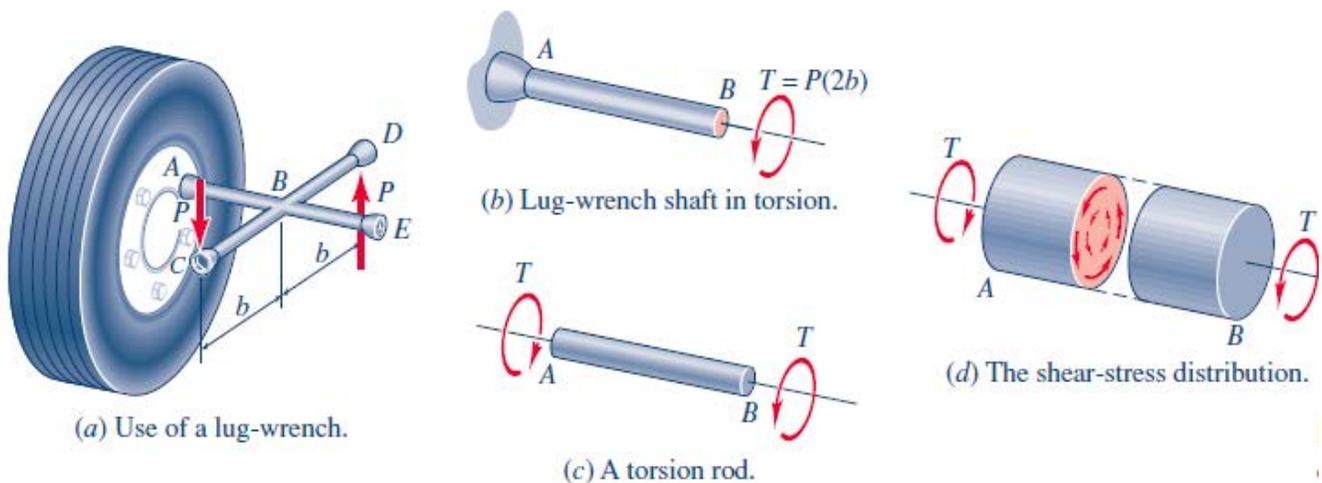


Figure (4.1): An example of torsion

- ▶ Several names are applied to torsion members, depending on the application:
 - *shaft*,
 - *torque tube*,
 - *torsion rod*,
 - *Torsion bar*, or
 - *Simply torsion member*.

❑ **Analogy Between Axial Deformation and Torsion**

- ▶ There is a direct analogy between axial deformation and torsion, as indicated by the entries in Table 4.1.

TABLE 4.1 Analogy Between Axial Deformation and Torsion	
Axial Deformation	Torsion
Axial Force (F)	Torque (T)
Elongation (e)	Twist angle (ϕ)
Normal stress (σ)	Shear stress (τ)
Extensional strain (ϵ)	Shear strain (γ)
Modulus of elasticity (E)	Shear modulus (G)

Sign convention

A **sign convention for torsion** is defined as follows:

- ▶ The longitudinal axis of the bar is labeled the x axis, with one end of the member being taken as the origin.
- ▶ A positive torque, $T(x)$, is a moment that acts on the cross section at x in a right-hand-rule sense about the outer normal to the cross section. On a cross sectional cut at x there will be equal and opposite torques $T(x)$, as indicated in Figure (4.2b).
- ▶ A positive angle of rotation, $\phi(x)$, is a rotation of the cross section at x in a **right-hand-rule sense** about the x axis, as illustrated in Figure (4.2c).

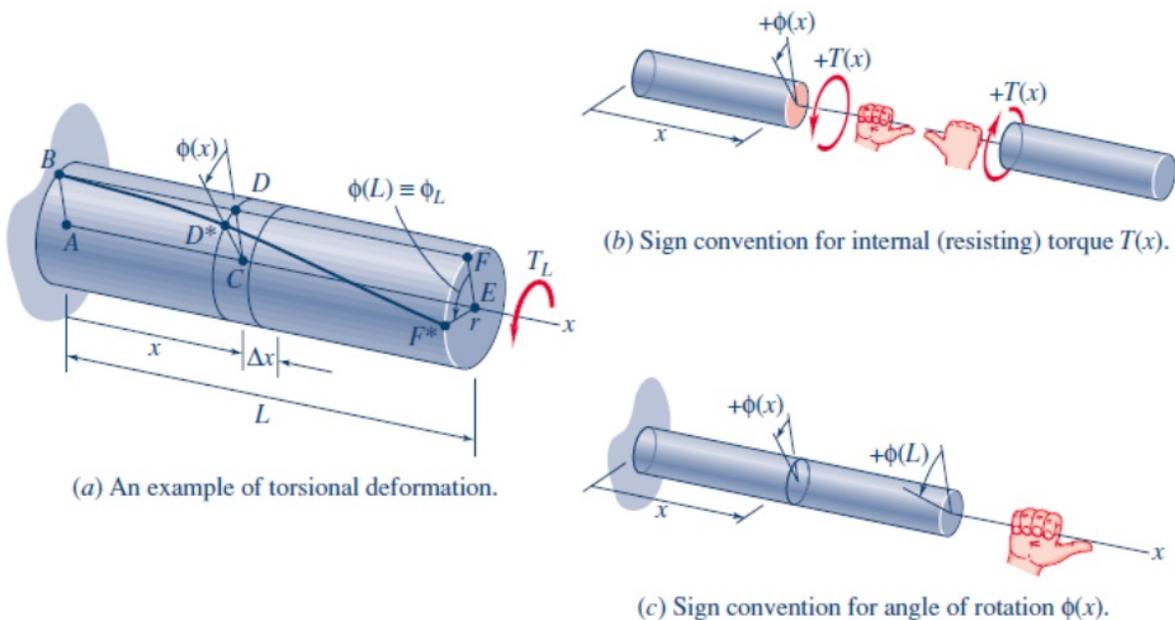


Figure (4.2): Torsional deformation; Sign convention for torsion

Assumptions

Following assumptions are made, while finding out shear stress in a circular shaft subjected to torsion:

- 1) The material of the shaft is uniform,
- 2) The twist along the shaft is uniform,
- 3) Normal cross – sections of the shaft, which were plane and circular before the twist, remain plane and circular even after the twist, and
- 4) All diameters of the normal cross – section, which were straight before the twist, remain straight with their magnitude unchanged, after the twist.

Strain-Displacement Analysis

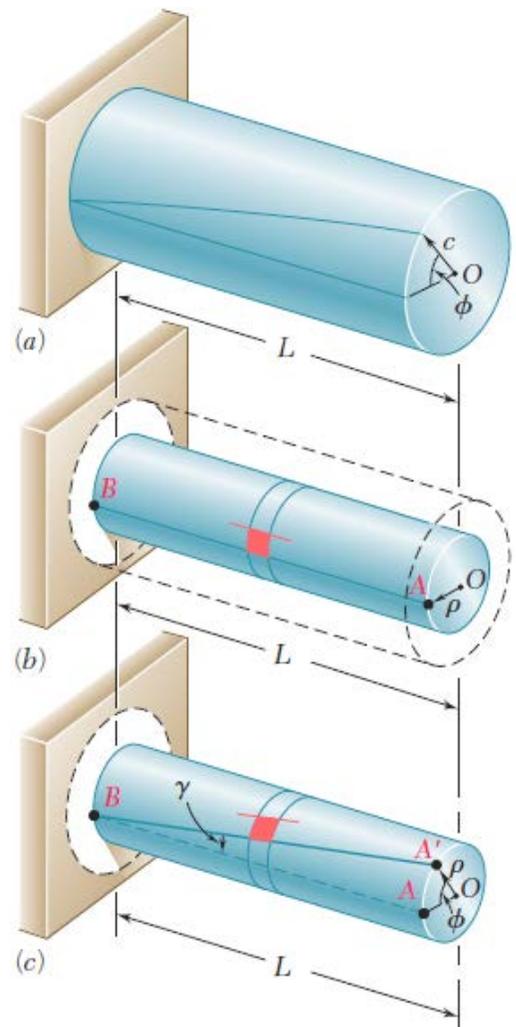
To determine the distribution of *shearing strains* in a circular shaft of length L and radius c that has been twisted through an angle ϕ (Figure 4.3a).

- ▶ Detaching from the shaft a cylinder of radius r , we consider the small square element before any load is applied (Figure 4.3b).
- ▶ As the shaft is subjected to a torsional load, the element deforms into a rhombus (Figure 4.3c).
- ▶ The shearing strain γ in a given element is measured by the change in the angles formed by the sides of that element.
- ▶ Since the circles defining two of the sides of the element considered here remain unchanged, the shearing strain γ must be equal to the angle between lines AB and A'B.
- ▶ We observe from Figure (4.3c) that, for small values of γ , we can express the arc length AA' as:

$$AA' = \gamma L$$

- ▶ But, on the other hand, we have:

$$AA' = \rho \phi$$



- ▶ It follows that:

$$\rho\phi = \gamma L$$

Or

$$\gamma = \frac{\rho\phi}{L} \quad \dots (4.1)$$

Where γ and ϕ are both expressed in radians.

- ▶ It follows from Eq. (4.1) that the shearing strain is maximum on the surface of the shaft, where $\rho = c$. We have:

$$\gamma_{max} = \frac{c \cdot \phi}{L} \quad \dots (4.2)$$

- ▶ Eliminating ϕ from Eqs. (4.1) and (4.2), we can express the shearing strain γ at a distance r from the axis of the shaft as:

$$\gamma = \frac{\rho}{c} \gamma_{max} \quad \dots (4.3)$$

STRESSES IN THE ELASTIC RANGE

- ▶ Let us now consider the case when the torque T is such that all shearing stresses in the shaft remain below the yield strength τ_y .
- ▶ Recalling Hooke's law for shearing stress and strain:

$$\tau = G\gamma \quad \dots (4.4)$$

Where G is the modulus of rigidity or shear modulus of the material.

Multiplying both members of Eq. (4.3) by G , we write:

$$G\gamma = \frac{\rho}{c} G\gamma_{max} \quad \dots (4.5)$$

$$\tau = \frac{\rho}{c} \tau_{max} \quad \dots (4.6)$$

- ▶ Figure (4.4a) shows the stress distribution in a solid circular shaft of radius c , and Figure (4.4b) in a hollow circular shaft of inner radius c_1 and outer radius c_2 . From Eq. (4.6), we find that, in the latter case,

$$\tau_{min} = \frac{c_1}{c_2} \tau_{max} \quad \dots (4.7)$$

- The sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude T of the torque exerted on the shaft:

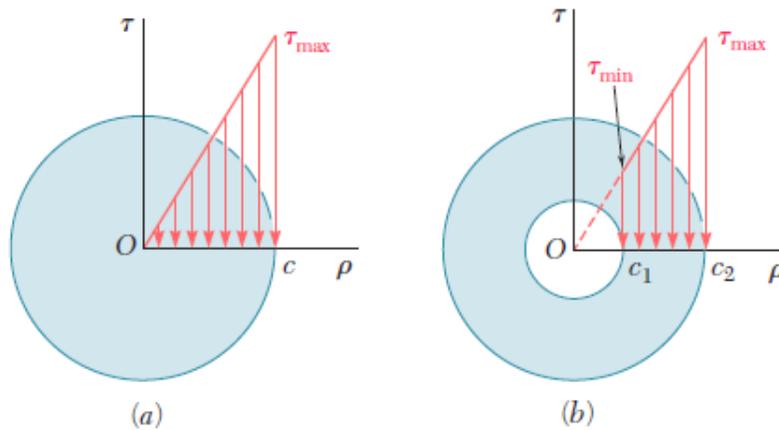


Figure (4.4): Distribution of shearing stresses.

$$\int \rho(\tau dA) = T \quad \dots (4.8)$$

Substituting for τ from (4.6) into (4.8), we write:

$$T = \int \rho(\tau dA) = \frac{\tau_{max}}{c} \int \rho^2 dA \quad \dots (4.9)$$

But, $\int \rho^2 dA = J$ (*Polar moment of unertia:*), then:

$$T = \frac{\tau_{max} \cdot J}{c} \quad \dots (4.10)$$

or

$$\tau_{max} = \frac{T \cdot c}{J} \quad \dots (4.11)$$

- Substituting for τ_{max} from Eqs.(4.11) into (4.6), we express the shearing stress at any distance ρ from the axis of the shaft as:

$$\tau = \frac{T\rho}{J} \quad \dots (4.12)$$

Equations (4.11) and (4.12) are known as *the elastic torsion formulas*.

- Substituting for (γ) from Eq. (4.1) into Eq. (4.4) and then substituting for (τ) from Eq. (4.4) into Eq.(4.12), we express the angle of twist (ϕ) in term of the torque (T):

$$\phi = \frac{Tl}{GJ} \quad \dots (4.13)$$

Recall from statics that the polar moment of inertia of a circle of radius (r) is:

$$J = \frac{1}{2} \pi r^4$$

- ▶ In the case of a hollow circular shaft of inner radius (c_1) and outer radius (c_2), the polar moment of inertia is:

$$J = \frac{1}{2} \pi c_2^4 - \frac{1}{2} \pi c_1^4 = \frac{1}{2} \pi (c_2^4 - c_1^4)$$

- ▶ In Eq. 4.13, the torque- T , G , and J may each vary with x as shown in Figure (4.5). This derivative form of the torque-twist equation may be integrated over the length of the member to give:

$$\phi = \int_0^L \frac{T(x) \cdot dx}{G(x) \cdot J(x)} \quad \dots (4.14)$$

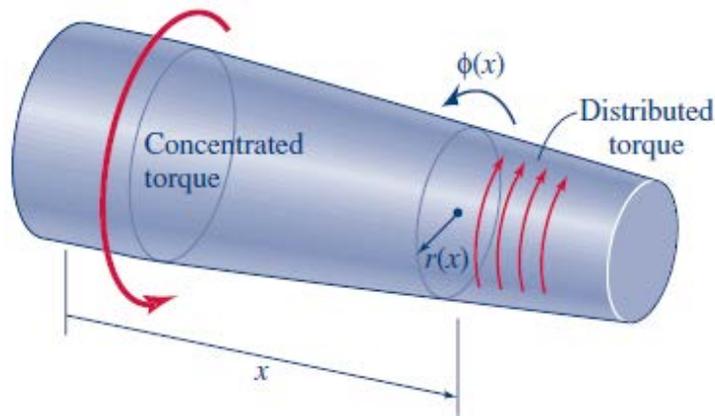


Figure (4.5)

Examples

Example (4.1): A uniform shaft of radius r and length L is subjected to a uniform distributed external torque t_0 (moment per unit length). (See Figure 4.6)

- Determine an expression for the maximum shear stress τ_{max} .
- Determine an expression for the total twist angle $\phi = \phi_L$.

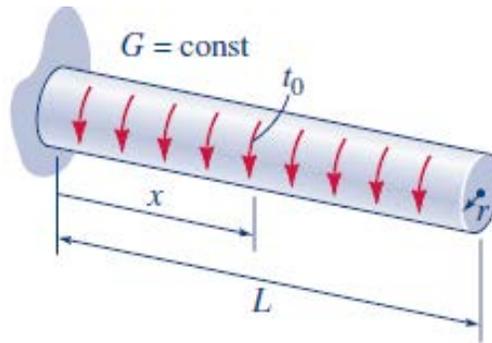


Figure (4.6)

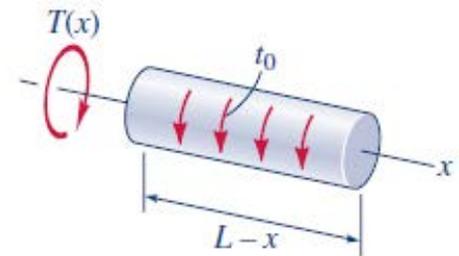
Solution:

a)

Equilibrium: On the section at x , the internal torque $T(x)$ in the positive sense according to the right-hand rule is:

$$\sum M_x = 0$$

$$T(x) = t_o(L - x)$$



Shear Stress: The maximum shear stress occurs at $x = 0$, where $T(0) = T_{max} = t_o(L)$. Then, from the torsion formula, Eq. 4.11,

$$\tau_{max} = \frac{t_o(L) \cdot r}{\frac{\pi}{2} r^4} = \frac{2t_o(L)}{\pi r^3}$$

b)

Torque-Twist: From the torque-twist relationship, Eq. 4.14,

$$\phi_L = \int_0^L \frac{t_o(L - x) \cdot dx}{G \cdot J} = \frac{t_o}{G \cdot J} \int_0^L (L - x) \cdot dx$$

$$\phi_L = \frac{t_o}{G \cdot J} \left[Lx - \frac{x^2}{2} \right]_0^L = \frac{t_o \cdot L^2}{2G \cdot J} = \frac{t_o \cdot L^2}{G \cdot \pi \cdot r^4}$$

Example (4.2): A hollow cylindrical steel shaft is **1.5 m** long and has inner and outer diameters respectively equal to **40** and **60 mm** (Figure 4.7).

(a) What is the largest torque that can be applied to the shaft if the shearing stress is not to exceed **120 MPa**?

(b) What is the corresponding minimum value of the shearing stress in the shaft?

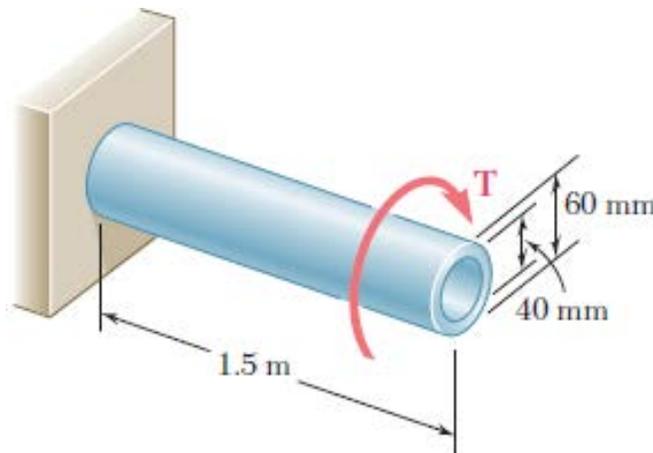


Figure (4.7)

Solution:

(a) Largest Permissible Torque.

The largest torque **T** that can be applied to the shaft is the torque for which $\tau_{max} = 120$ MPa. Since this value is less than the yield strength for steel, we can use Eq. (4.11). Solving this equation for **T**, we have:

$$\tau_{max} = \frac{T \cdot c}{J} \rightarrow T = \frac{\tau_{max} \cdot J}{c}$$

$$J = \frac{1}{2} \pi c_2^4 - \frac{1}{2} \pi c_1^4 = \frac{1}{2} \pi (c_2^4 - c_1^4)$$

$$c_1 = \frac{40}{2} = 20 \text{ mm (or 0.02 m)}$$

$$c_2 = \frac{60}{2} = 30 \text{ mm (or 0.03 m)}$$

$$J = \frac{1}{2} \pi ((0.03)^4 - (0.02)^4) = 1.021 \times 10^{-6} \text{ m}^4$$

$$T = \frac{\tau_{max} \cdot J}{c} = \frac{(120 \times 10^6 \text{ Pa})(1.021 \times 10^{-6} \text{ m}^4)}{0.03} = 4.08 \text{ kN.m}$$

(b) Minimum Shearing Stress. The minimum value of the shearing stress occurs on the inner surface of the shaft. It is obtained from Eq. (4.7), which expresses that τ_{min} and τ_{max} are respectively proportional to c_1 and c_2 :

$$\tau_{min} = \frac{c_1}{c_2} \tau_{max}$$
$$\tau_{min} = \frac{c_1}{c_2} \tau_{max} = \frac{0.02}{0.03} (120) = 80 \text{ MPa}$$

Example (4.3): What is the minimum diameter of a solid steel shaft that will not twist through more than (3°) in a (6 m) length when subjected to a torque of (**14 kN.m**)? What maximum shearing stress is developed? Use **G = 83 GPa**.

Solution:

$$\phi = \frac{Tl}{GJ} \rightarrow 3 \left(\frac{\pi}{180} \right) = \frac{(14 \times 10^3)(6)}{(83 \times 10^9) \left(\frac{\pi}{2} r^4 \right)}$$
$$\rightarrow r = 59.23 \text{ mm or } d = 118.5 \text{ mm}$$
$$\tau_{max} = \frac{Tr}{J} = \frac{14 \times 10^6(59.23)}{\frac{\pi}{2} (59.23)^4} = 42.9 \text{ MPa}$$

Example (4.4): Determine the length of the shortest (**2mm**) diameter bronze wire which can be twisted through two complete turns without exceeding a shearing stress of (**70 MPa**). Use **G = 83 GPa**.

Solution:

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (2)^4 = 1.571 \text{ mm}^4$$
$$\tau_{max} = \frac{T \cdot c}{J} \rightarrow T = \frac{\tau_{max} \cdot J}{c} \rightarrow$$
$$T = \frac{70 (\pi(1)^4)}{2(1)} \rightarrow T \cong 110 \text{ N.mm}$$

$$\phi = \frac{Tl}{GJ} \rightarrow 720 \left(\frac{\pi}{180} \right) = \frac{(110)(l)}{(83 \times 10^3)(1.571)}$$

$$\rightarrow l = 14895 \text{ mm or } 14.895 \text{ m}$$

Example (4.5): A compound shaft consisting of an aluminum segment and a steel segment is acted upon by two torque as shown in Figure (4.8). Determine the maximum permissible value of (T) subjected to the following conditions ($\tau_s \leq 100 \text{ MPa}$, $\tau_{al} \leq 70 \text{ MPa}$, and the angle of rotation of the free end limited to (12°)). Use $G_s = 83 \text{ GPa}$ and $G_{al} = 28 \text{ GPa}$.

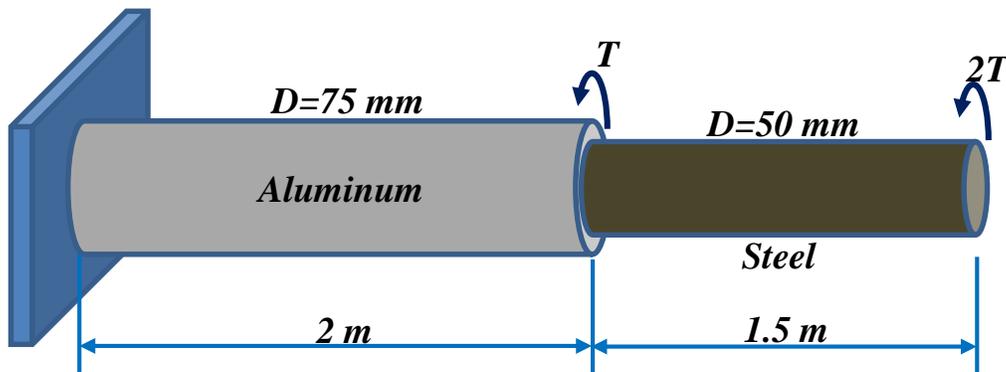


Figure (4.8)

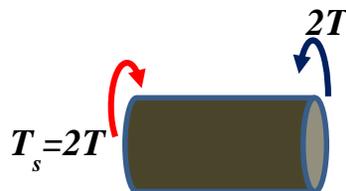
Solution:

$$\tau_s = \frac{T \cdot c}{J} \rightarrow T = \frac{\tau_s \cdot J}{c} \rightarrow T_s = \frac{\tau_s (\pi c^4)}{2 \cdot c} = \frac{\tau_s (\pi c^3)}{2} = \frac{100 (\pi (25)^3)}{2} \times 10^{-3}$$

$$= 2454.37 \text{ N.m}$$

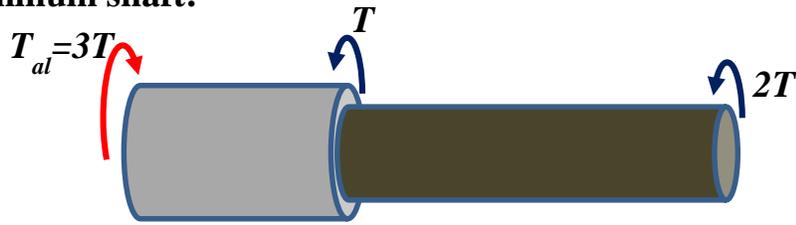
$$T_{al} = \frac{\tau_s (\pi c^3)}{2} = \frac{70 (\pi (37.5)^3)}{2} \times 10^{-3} = 5798.45 \text{ N.m}$$

Section in steel shaft:



$$T_s = 2T \rightarrow T = \frac{T_s}{2} = \frac{2454.37}{2} = 1227.2 \text{ N.m}$$

Section in aluminum shaft:



$$T_{al} = 3T \rightarrow T = \frac{T_{al}}{3} = \frac{5798.45}{3} = 1932.82 \text{ N.m}$$

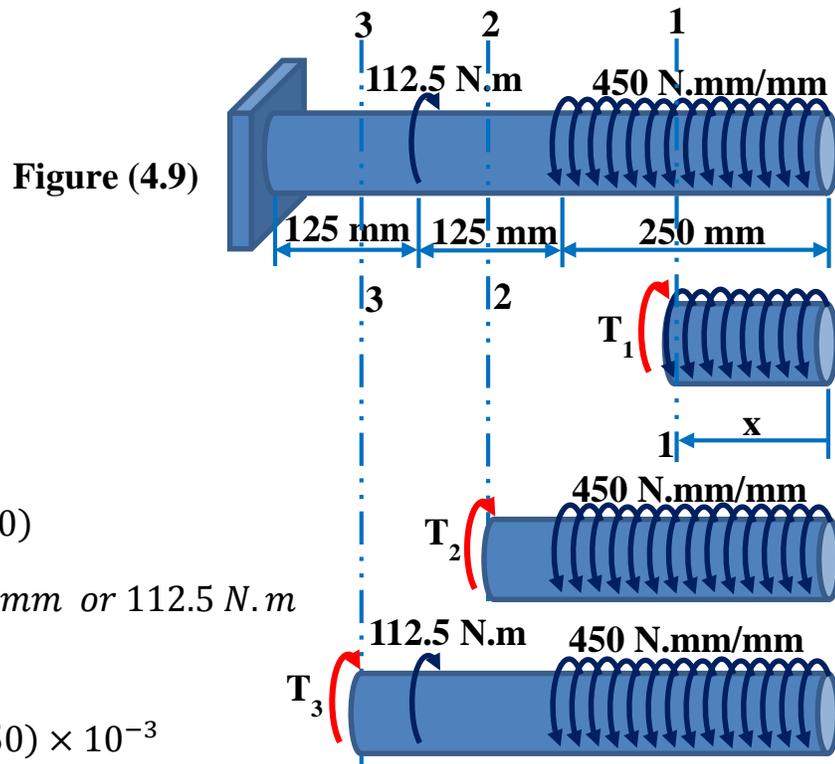
$$\phi_{free} = \phi_s + \phi_{al}$$

$$12 \left(\frac{\pi}{180} \right) = \frac{2T(1.5)}{(83 \times 10^9) \frac{\pi}{32} (50)^4 \times 10^{-12}} + \frac{3T(2)}{(83 \times 10^9) \frac{\pi}{32} (75)^4 \times 10^{-12}}$$

$$12 = 7.328 \times 10^{-3} T \rightarrow T = 1637.53 \text{ N.m}$$

Use $T = 1227.2 \text{ N.m}$

Example (4.6): For the shaft shown in Figure (4.9), find the maximum shear stress and total angle of rotation. Assume $G=83 \text{ GPa}$ and diameter of shaft (25 mm).



Solution:

Sec.(1-1) as F.B.D.

$$T = 450x$$

Sec.(2-2) as F.B.D.

$$T = 450(250)$$

$$T = 112500 \text{ N.mm or } 112.5 \text{ N.m}$$

Sec.(3-3) as F.B.D.

$$T + 112.5 = 450(250) \times 10^{-3}$$

$$\rightarrow T = 0$$

$$\therefore T = 112.5 \text{ N.m} \quad (\text{Control})$$

$$\therefore \tau_{max} = \frac{T \cdot c}{J} = \frac{(112.5 \times 10^3)(12.5)}{\frac{\pi}{2}(12.5)^4} = 36.67 \text{ MPa}$$

$$\phi_{total} = \sum \phi_i = 0 + \phi_1 + \phi_2$$

$$\phi_1 = \int_0^{250} \frac{(450x)dx}{GJ} = \frac{450x^2}{2GJ} \Big|_0^{250} = \frac{450(250)^2}{2(84 \times 10^3) \left(\frac{\pi}{32}(25)^4\right)}$$

$$\phi_1 = 4.365 \times 10^{-3} \text{ rad.}$$

$$\phi_2 = \frac{112.5 \times 10^3(125)}{(84 \times 10^3) \left(\frac{\pi}{32}(25)^4\right)} = 4.365 \times 10^{-3} \text{ rad.}$$

$$\phi_{total} = [4.365 \times 10^{-3} + 4.365 \times 10^{-3}] \frac{180}{\pi} = 0.5^\circ$$

Example (4.7): Find the total angle of rotation for the prismatic shaft shown in Figure (4.10), which is subjected to distributed torsional moment ($T_{(x)}=kx \text{ N.mm/mm}$), GJ and k are constants.

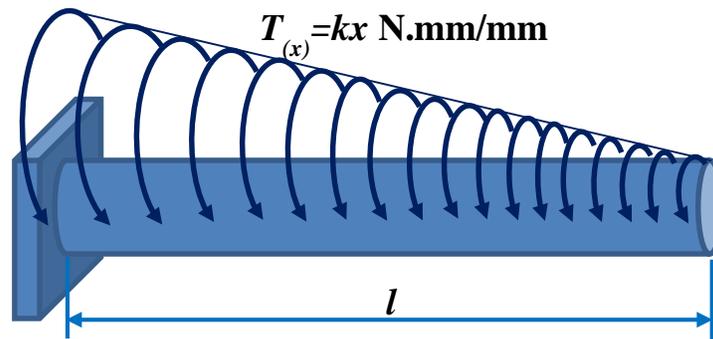
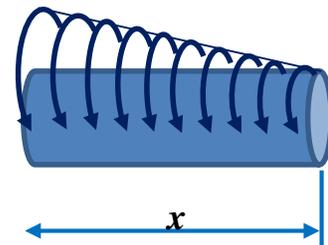


Figure (4.10)

Solution:

$$T = \int_0^x (kx)dx = \frac{kx^2}{2}$$



$$\phi = \int_0^l \frac{T}{GJ} dx = \frac{1}{GJ} \int_0^l \frac{kx^2}{2} dx$$

$$\therefore \phi = \frac{1}{2GJ} \left[\frac{kx^3}{3} \right]_0^l = \frac{kl^3}{6GJ}$$

To find max. Shear stress:

$$\tau_{max} \frac{16T}{\pi d^3} = \frac{16kx^2}{2\pi d^3} \Bigg|_{x=l} = \frac{16kl^2}{2\pi d^3} = \frac{8kl^2}{\pi d^3}$$

Example (4.8): For the non – prismatic shaft shown in Figure (4.11). Find the total rotation.

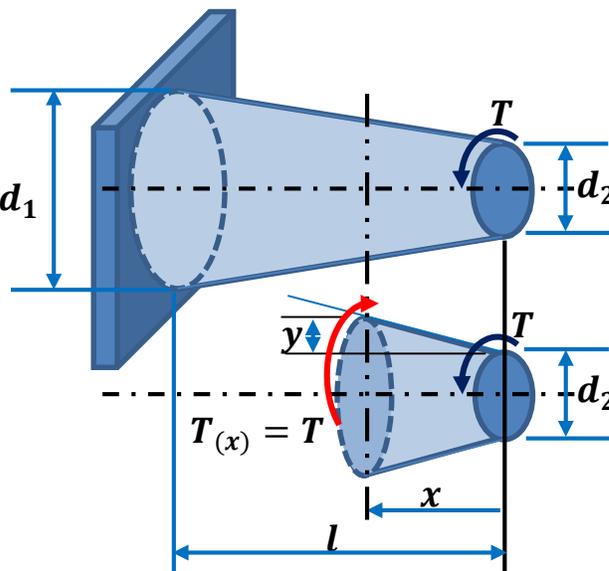
Solution:

$$\frac{\frac{d_1 - d_2}{2}}{l} = \frac{y}{x}$$

$$\rightarrow y = \frac{d_1 - d_2}{2l} x$$

$$\phi = \int_0^l \frac{T(x) dx}{GJ(x)}$$

Figure (4.11)



$$J(x) = \frac{\pi}{32} (D(x))^4$$

$$D(x) = 2y + d_2 = \frac{d_1 - d_2}{l} x + d_2$$

$$J(x) = \frac{\pi}{32} \left[\frac{d_1 - d_2}{l} x + d_2 \right]^4$$

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$$\phi = \int_0^l \frac{T dx}{G \frac{\pi}{32} \left[\frac{d_1 - d_2}{l} x + d_2 \right]^4} = \frac{32T}{G\pi} \int_0^l \left[\frac{d_1 - d_2}{l} x + d_2 \right]^{-4} dx$$

Multiply by $\frac{\frac{d_1 - d_2}{l}}{\frac{d_1 - d_2}{l}}$, get:

$$\begin{aligned} \phi &= \frac{32T}{G\pi(d_1 - d_2)} \left[\left[\frac{d_1 - d_2}{l} x + d_2 \right]^{-3} \cdot \frac{1}{-3} \right]_0^l \\ &= \frac{32T}{-3G\pi(d_1 - d_2)} \left[(d_1)^{-3} - (d_2)^{-3} \right] \\ \phi &= \frac{32T}{3G\pi} \left[\frac{(d_1)^2 + d_1 d_2 + (d_2)^2}{(d_1)^3 (d_2)^3} \right] \end{aligned}$$