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Areas, Moments and Center of Mass :-

The area of a closed bounded plane region R is:

$$A = \iint_R dA = \iint_R dx\,dy = \iint_R dy\,dx$$

To find the moments and center of mass of thin sheet or plate we use the formulas :-

(1) Mass :- $M = \iint \delta(x,y) \cdot dA$

where $\delta(x,y)$ is the density

(2) First moments :-

$$M_x = \iint y \cdot \delta(x,y) \cdot dA$$

$$M_y = \iint x \cdot \delta(x,y) \cdot dA$$

(3) Center of mass :-

$$\bar{x} = M_y / M$$

$$\bar{y} = M_x / M$$

(4) Moment of inertia :-

- About x-axis $I_x = \iint y^2 \cdot \delta(x,y) \cdot dA$
- About y-axis $I_y = \iint x^2 \cdot \delta(x,y) \cdot dA$
- About the origin $I_o = \iint (x^2+y^2) \cdot \delta(x,y) \cdot dA$

(5) Radii of gyration :-

- About x-axis $R_x = \sqrt{I_x / M}$
- About y-axis $R_y = \sqrt{I_y / M}$
- About the origin $R_o = \sqrt{I_o / M}$

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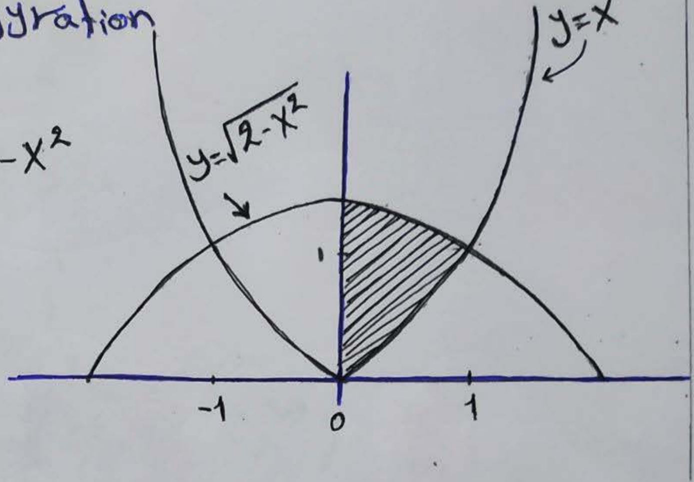
EX:- A thin plate covers the plane region R bounded by $y = x^2$, $y = \sqrt{2-x^2}$ and y-axis and with a density function $\delta = xy$

a- Find the mass of the body
b- Find the center of the mass
c- Find the radii of gyration



Solution:-

$$x^2 = \sqrt{2-x^2} \Rightarrow x^4 = 2 - x^2$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2+2)(x^2-1) = 0$$

$$x^2 - 1 = 0 \Rightarrow x^2 = 1$$

$$\therefore \boxed{x = \mp 1}$$

$$x^2 + 2 = 0 \Rightarrow x^2 = -2 \quad \text{مهمل}$$

$$\therefore \boxed{y = 1^2 = 1}$$

a/

$$M = \iint_R \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} xy \cdot dy\, dx$$

$$= \int_0^1 x \frac{y^2}{2} \Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 x \left[\left(\frac{(\sqrt{2-x^2})^2}{2} \right) - \frac{(x^2)^2}{2} \right] dx$$

$$= \int_0^1 x \cdot \left[\frac{2-x^2}{2} - \frac{x^4}{2} \right] dx$$

$$= \int_0^1 \left[x - \frac{x^3}{2} - \frac{x^5}{2} \right] dx = \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{12} \Big]_0^1$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12} = \boxed{\frac{7}{24}}$$

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b/ $$M_x = \iint_R y\cdot\delta(x,y)\cdot dA = \int_0^1\int_{x^2}^{\sqrt{2-x^2}} x y^2\cdot dy\, dx$$

$$= \int_0^1 x\,\frac{y^3}{3}\Big]_{x^2}^{\sqrt{2-x^2}}\cdot dx$$

$$= \int_0^1 x\cdot\left[\left(\frac{(\sqrt{2-x^2})^3}{3}\right) - \frac{(x^2)^3}{3}\right]\cdot dx$$

$$= \int_0^1 \left[\frac{x(2-x^2)^{3/2}}{3} - \frac{x^7}{3}\right]\cdot dx$$

$$= \frac{-1}{2*3}\cdot\frac{(2-x^2)^{5/2}}{5/2} - \frac{x^8}{8*3}\Big]_0^1 =$$

$$= \left[\left(\frac{-1}{15}(2-1)^{5/2} - \frac{1^8}{24}\right)\right] - \left[\left(\frac{-1}{15}(2-0^2)^{5/2}\right) - \frac{0^8}{24}\right]$$

$$= \boxed{0.268}$$

$$\therefore\ \bar{y} = M_x/M = \frac{0.268}{7/24} = \boxed{0.922}$$

$$M_y = \iint_R x\cdot\delta(x,y)\, dA = \int_0^1\int_{x^2}^{\sqrt{2-x^2}} x^2 y\cdot dy\cdot dx$$

$$= \int_0^1 x^2\,\frac{y^2}{2}\Big]_{x^2}^{\sqrt{2-x^2}}\cdot dx = \int_0^1 x^2\left[\frac{(\sqrt{2-x^2})^2}{2} - \frac{(x^2)^2}{2}\right]\cdot dx$$

$$= \int_0^1 x^2\left[\frac{2-x^2}{2} - \frac{x^4}{2}\right]\cdot dx = \int_0^1 \left(x^2 - \frac{x^4}{2} - \frac{x^6}{2}\right)\cdot dx$$

$$= \left(\frac{x^3}{3} - \frac{x^5}{10} - \frac{x^7}{14}\right]_0^1 = \frac{1}{3} - \frac{1}{10} - \frac{1}{14} = \boxed{\frac{17}{105}}$$

$$\therefore\ \bar{x} = M_y/M = \frac{17/105}{7/24} = \boxed{0.56}$$

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c/ $I_x = \iint_R y^2 \cdot \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x y^3 \cdot dy \cdot dx$

$= \int_0^1 x \frac{y^4}{4}\Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 \frac{x}{4}\left[(\sqrt{2-x^2})^4 - (x^2)^4\right] \cdot dx$

$= \int_0^1 \left[\frac{x}{4}(2-x^2)^2 - \frac{x^9}{4}\right] dx = \frac{-1}{4*2}\frac{(2-x^2)^3}{3} - \frac{x^{10}}{40}\Big]_0^1$

$= \left[\frac{-1}{24}(2-1^2)^3 - \frac{1}{40}\right] - \left[\frac{-1}{24}(2-0)^3 - \frac{0}{40}\right]$

$= \frac{-1}{24} - \frac{1}{40} + \frac{8}{24} = \frac{4}{15} = \boxed{0.27}$

$\therefore R_x = \sqrt{I_x/M} = \sqrt{\frac{4/15}{7/24}} = \boxed{0.96}$

$I_y = \iint_R x^2 \cdot \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x^3 y \cdot dy \cdot dx$

$= \int_0^1 x^3 \frac{y^2}{2}\Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 \frac{x^3}{2}\left[(\sqrt{2-x^2})^2 - (x^2)^2\right] \cdot dx$

$= \int_0^1 \left[x^3 - \frac{x^5}{2} - \frac{x^7}{2}\right] dx = \frac{x^4}{4} - \frac{x^6}{12} - \frac{x^8}{16}\Big]_0^1$

$= \frac{1}{4} - \frac{1}{12} - \frac{1}{16} = \boxed{\frac{5}{48}}$

$\therefore R_y = \sqrt{I_y/M} = \sqrt{\frac{5/48}{7/24}} = \boxed{0.597}$

$I_o = I_x + I_y = \frac{4}{15} + \frac{5}{48} = \boxed{\frac{89}{240}}$

$R_o = \sqrt{I_o/M} = \sqrt{\frac{89/240}{7/24}} = \boxed{1.13}$