M.Sc. Murtadha Al-Masoudy
Math. 2.
2nd Stage.

Infinite Sequences and Series

* Sequenses of Numbers :-

A Sequence of number $\{a_n\}$ is a function whose domain is the set of positive integers.

For example

a_n	Sequence
$a_n = n-1$	$0, 1, 2, ---, n-1$
$a_n = 1/n$	$1, \frac{1}{2}, \frac{1}{3}, ---, \frac{1}{n}$
$a_n = (-1)^{n+1}\frac{1}{n}$	$1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, ---, (-1)^{n+1}\frac{1}{n}$
$a_n = 3$	$3, 3, 3, ----, 3$

Convergence and Divergence :-

The Sequence $\{a_n\}$ Converges to the number L if to every Positive number ϵ there Corresponds an index N such that

$$|a_n - L| < \epsilon \qquad \text{for all } n > N$$

If not such limit exists, we Say that $\{a_n\}$ diverges

If $\{a_n\}$ Converges then its limit is unique.

$\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$

L is called the limit of Sequence a_n

Lecturer: M.Sc Murtadha Mohsen Al-Masoudy
E-mail: Murtadha_Almasoody@mustaqbal-college.edu.iq

Sequences

① $\lim\limits_{n \to \infty} (n)^{1/n} = 1$

② $\lim\limits_{n \to \infty} \frac{\ln n}{n} = 0$ (L-hopital role).

③ $\lim\limits_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

④ $\lim\limits_{n \to \infty} r^n = 0 \;;\; |r| < 1$.

⑤ $\lim\limits_{n \to \infty} r^{1/n} = 1 \;;\; r > 0$

⑥ $\lim\limits_{n \to \infty} \frac{r^n}{n!} = 0$

⑦ $\lim\limits_{n \to \infty} \frac{n!}{(n)^n} = 0$

$\{a_n\}$	Sequence
n^2	$1, 4, 9, 16, \ldots$
$2n^2$	$2, 8, 18, 32, \ldots$
2^n	$2, 4, 8, 16, \ldots$
$3n+2$	$5, 8, 11, 14, \ldots$
$2n^2-1$	$1, 7, 17, 31, \ldots$

Ex:- Determine the convergence of:

(1) $\{a_n\}_{n=1}^{\infty} = \left\{\frac{n+1}{n}\right\}$ (2) $\left\{\frac{2n^2}{3n^2+n+1}\right\}$

(3) $\left\{\frac{1}{3^n}\right\}$ (4) $\left\{\frac{1}{n}\right\}$ (5) $\{3\}$

Solution:-

(1) $\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$

(2) $\lim_{n\to\infty}\left(\frac{2n^2}{3n^2+n+1}\right) = \lim_{n\to\infty} \frac{2\cancel{n^2}}{\cancel{n^2}\left(3+\frac{1}{n}+\frac{1}{n^2}\right)}$

$= \lim_{n\to\infty}\left(\frac{2}{3+\frac{1}{n}+\frac{1}{n^2}}\right) = \frac{2}{3}$

(3) $\lim_{n\to\infty}\left(\frac{1}{3^n}\right) = \frac{1}{\infty} = 0$

(4) $\lim_{n\to\infty}\left(\frac{1}{n}\right) = \frac{1}{\infty} = 0$

(5) $\lim_{n\to\infty}(3) = 3$

Note:-

$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

L'Hopital Rule :-

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{f'(x)}{g'(x)} = \frac{f'(\infty)}{g'(\infty)}$$

Ex:- Find by using L'Hopital rule:

1. $\lim_{n\to\infty} \frac{\ln n}{n}$
2. $\lim_{n\to\infty} \frac{2^n}{5n}$
3. $\lim_{n\to\infty} \sqrt[n]{n}$
4. $\lim_{n\to\infty} x^{1/n}$

Solution:-

1. $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = \frac{0}{1} = 0$

2. $\lim_{n\to\infty} \frac{2^n}{5n} = \lim_{n\to\infty} \frac{2^n \ln 2}{5} = \frac{\infty \ln 2}{5} = \infty$

3. $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} (n)^{1/n} = \lim_{n\to\infty} e^{\ln(n)^{1/n}}$

$$= \lim_{n\to\infty} e^{1/n \ln n} = e^0 = 1$$

4. $\lim_{n\to\infty} x^{1/n} = \lim_{n\to\infty} e^{\ln(x)^{1/n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln x}$

$$= e^0 = 1$$

* Infinite Series:-

Given a sequence of numbers $\{a_n\}$ an expression of the form $a_1 + a_2 + a_3 + \dots + a_n + \dots$ is called an infinite series. The number a_n is called the nth term of series.

The sequence $\{S_n\}$ defined by:

$$S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^{n} a_k$$

The sequence $\{S_n\}$ is called the sequence of partial sums of the series. If this sequence of partial sums converges to a limit (L), the series converges and that its sum is (L).

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, then the series diverges.

* Difinition of Convergence of a Series:-

The series $\sum_{n=1}^{\infty} a_n$ converges to (S) means that

$\lim_{n\to\infty} S_n = S$ where S is a single finite number.

Ex:- Find (S_n) by use the difinition to determine whether the series converges or diverges.

① $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ ② $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$

Solution:-

① $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) \Rightarrow a_n = \ln\left(\frac{n}{n+1}\right) = \ln n - \ln(n+1)$

$a_1 = \ln 1 - \ln(1+1) = \ln 1 - \cancel{\ln 2}$

$a_2 = \ln 2 - \ln(2+1) = \cancel{\ln 2} - \cancel{\ln 3}$

$a_3 = \ln 3 - \ln(3+1) = \cancel{\ln 3} - \ln 4$

$\vdots$

$a_{n-1} = \ln(n-1) - \ln(n-1+1) = \ln(n-1) - \cancel{\ln n}$

$a_n = \cancel{\ln n} - \ln(n+1)$

$\therefore S_n = \ln 1 - \ln(n+1) = -\ln(n+1)$

$\because \lim_{n\to\infty} S_n = \lim_{n\to\infty}(-\ln(n+1))$ is not a finite number

$\therefore \sum \ln\left(\frac{n}{n+1}\right)$ is divergence.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)$

$= \lim_{n\to\infty} \ln\left(\frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}\right)$

$= \lim_{n\to\infty} \ln\left(\frac{1}{1+\frac{1}{n}}\right) = \ln 1 = 0$

Lecturer: M.Sc Murtadha Mohsen Al-Masoudy
E-mail: Murtadha_Almasoody@mustaqbal-college.edu.iq

(2) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$

$$a_n = \frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3} = \frac{A(n+3)+B(n+1)}{(n+1)(n+3)}$$

$$1 = A(n+3) + B(n+1)$$

$$1 = An + 3A + Bn + B$$

$$n^1,\ 0 = A + B \quad —(1)$$

$$n^0,\ -1 = -3A - B \quad —(2)$$

بالطرح

$$-1 = -2A \Rightarrow \boxed{A = 1/2}$$

نعوض قيمة A في معادلة (1)

$$0 = \frac{1}{2} + B \Rightarrow \boxed{B = -1/2}$$

$$\therefore a_n = \frac{1/2}{n+1} + \frac{-1/2}{n+3} = \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+3}\right]$$

$$a_1 = \frac{1}{2}\left[\frac{1}{1+1} - \frac{1}{1+3}\right] = \frac{1}{2}\left[\frac{1}{2} - \cancel{\frac{1}{4}}\right]$$

$$a_2 = \frac{1}{2}\left[\frac{1}{2+1} - \frac{1}{2+3}\right] = \frac{1}{2}\left[\frac{1}{3} - \cancel{\frac{1}{5}}\right]$$

$$a_3 = \frac{1}{2}\left[\frac{1}{3+1} - \frac{1}{3+3}\right] = \frac{1}{2}\left[\cancel{\frac{1}{4}} - \frac{1}{6}\right]$$

$$a_4 = \frac{1}{2}\left[\frac{1}{4+1} - \frac{1}{4+3}\right] = \frac{1}{2}\left[\cancel{\frac{1}{5}} - \frac{1}{7}\right]$$

$$\vdots$$

$$a_{n-2} = \frac{1}{2}\left[\frac{1}{n-2+1} - \frac{1}{n-2+3}\right] = \frac{1}{2}\left[\frac{1}{n-1} - \cancel{\frac{1}{n+1}}\right]$$

$$a_{n-1} = \frac{1}{2}\left[\frac{1}{n-1+1} - \frac{1}{n-1+3}\right] = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

$$a_n = \frac{1}{2}\left[\cancel{\frac{1}{n+1}} - \frac{1}{n+3}\right]$$

Lecturer: M.Sc Murtadha Mohsen Al-Masoudy
E-mail: Murtadha_Almasoody@mustaqbal-college.edu.iq

$$\therefore S_n = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3}\right]$$

$$= \boxed{\frac{5}{12}}$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$ Converges to $\frac{5}{12}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(n+1)(n+3)}$$

$$= 0$$

م.م مرتضى محسن المسعودي