Lecturer: M.Sc Murtadha Mohsen Al-Masoudy
E-mail: Murtadha_Almasoody@mustaqbal-college.edu.iq

Series

م.م. مرتضى المسعودي

① +ve terms series $+ + + \cdots$

② alternating series $+ - + - + \cdots$

③ Power series

special types

A) Geometric series.

$a(1+r+r^2+r^3+\cdots)$ Conv. if $|r|<1$

$$S = \frac{a}{1-r}$$

B) P- Series

$$\sum_{n=1}^{\infty} \frac{c}{n^p}$$ Conv. if $P>1$

* If $P=1$, the series is Called (Harmonic Series) which is Divergent.

Conv. Div. Test

(A) Test on +ve terms series $\sum_{n=k}^{\infty} a_n$

① Divergence (n^{th} term test)

$$\lim_{n\to\infty} a_n \neq \text{zero}$$

- لمعرفة ان المتوالية متباعدة فقط.
واذا كانت النتيجة مساوية الى الصفر
فهذا لا يعني انها متقاربة.

Ex: Determine whether the series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$$

1

Sol:-

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} \quad \div n^2 \text{ (greatest Power of } n).$$

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} \Rightarrow \frac{1}{1+0} = 1 \neq \text{Zero}$$

$\therefore$ Div. by Div. test.

② Integral (Mclaurin-Cauchy test)

$$\sum_{c}^{\infty} a_n = \int_{c}^{\infty} a(x) \cdot dx.$$

- شرط ان تكون المتوالية $a(x)$ متناقصة بالفترة $(c-\infty)$
- اذا كان ناتج التكامل قيمة فيكون Con. ، عدا ذلك Div.
- يطبق هنا الاختبار اذا كان التكامل سهلاً.

Ex:

$$\sum_{K=2}^{\infty} \frac{1}{K(\ln K)^2} - \frac{1}{K^2}$$

Sol.

$$\int_2^\infty \frac{1}{x(\ln x)^2} \cdot dx = \int_2^\infty \frac{(\ln x)^{-2}}{x} \cdot dx = -\frac{1}{\ln x}\Big]_2^\infty$$

$$= -\left(\frac{1}{\ln\infty} - \frac{1}{\ln 2}\right)$$

$$= 1/\ln 2$$

$\therefore$ Conv. by Integral test.

$$\int_2^\infty \frac{1}{x^2} \cdot dx = \int_2^\infty x^{-2} \cdot dx = \frac{-1}{x}\Big]_2^\infty$$

$$= \frac{1}{2}$$ $\therefore$ Conv. by Integral test.

$$\therefore S = \left[\left(\frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \frac{1}{4(\ln 4)^2} + \cdots\right)\right] - \left[\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right]$$

③ Ratio $\quad \rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

- تستخدم هذه الطريقة عادةً عندما تحتوي المتوالية على مضروب او قوة.

$\rho < 1 \Rightarrow$ Conv.
$\rho > 1 \Rightarrow$ Div.
$\rho = 1 \Rightarrow$ Fails.

2

Lecturer: M.Sc Murtadha Mohsen Al-Masoudy
E-mail: Murtadha_Almasoody@mustaqbal-college.edu.iq

Ex:-

$$\sum_{n=0}^{\infty} \frac{(n+4)!}{4!\,n!\,4^n}$$

Sol.

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{(n+5)!/4!\cdot(n+1)!\cdot 4^{n+1}}{(n+4)!/4!\,n!\,4^n}$$

$$= \frac{(n+5)!\,\cancel{4!}\,n!\,\cancel{4^n}}{(n+4)!\,\cancel{4!}\,(n+1)!\,\cancel{4^{n+1}}\,4}$$

$$= \frac{(n+5)\cancel{(n+4)!}\,n!}{\cancel{(n+4)!}\,(n+1)!\,4} = \frac{(n+5)\,\cancel{n!}}{(n+1)\,\cancel{n!}\,4}$$

$$= \left(\frac{1}{4}\right)\frac{n+5}{n+1} \quad \div n$$

$$= \frac{1}{4}\left(\frac{1+\frac{5}{n}}{1+\frac{1}{n}}\right)$$

$$= \frac{1}{4}\left(\frac{1+0}{1+0}\right)$$

$$= \frac{1}{4} < 1$$

$\therefore$ Conv. by Ratio test.

$$\therefore S = \frac{1}{4!}\left(\frac{4!}{0!\,(1)} + \frac{5!}{1!\,(4)} + \frac{6!}{2!\,(4)^2} + \cdots\right).$$

④ Root

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{1/n}$$

- تستخدم عندما تكون المتوالية مرفوعة الى قوة بحيث $\frac{1}{n}$ تزيلها.

$$\rho < 1 \Rightarrow \text{Conv.}$$

$$\rho > 1 \Rightarrow \text{Div.}$$

$$\rho = 1 \Rightarrow \text{Fails.}$$

Ex:-

$$\sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{4^n}$$

Sol.

$$\rho = \lim_{n\to\infty} \left(\frac{1}{2^{\cancel{n}}}\right)^{1/\cancel{n}} = \frac{1}{2} < 1.$$

3

Lecturer: M.Sc Murtadha Mohsen Al-Masoudy
E-mail: Murtadha_Almasoody@mustaqbal-college.edu.iq

$$\rho = \lim_{n\to\infty} \left(\frac{1}{4^n}\right)^{1/n} = \frac{1}{4} < 1 \Rightarrow \therefore \text{ Conv. by root test.}$$

$$S = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) + \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right).$$

$$S = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) + \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right).$$

Geometric $r_1 = \frac{1}{2}$, $r_2 = \frac{1}{4}$

$$S = \frac{1/2}{1/2} + \frac{1}{3} = \frac{4}{3}$$

⑤ Comparison

$a_n \leq C_n$ — اذا كانت C_n متقاربة، فان a_n متقاربة ايضاً.

$a_n \geq D_n$ — اذا كانت d_n متباعدة، فان a_n متباعدة ايضاً.

- يتم اختيار C_n , d_n عن طريق قسمة اكبر اس لـ n في البسط على اكبر اس لـ n في المقام.

⑥ Limit Comparison.

- يستخدم كثيراً

$$\lim_{n\to\infty} \frac{b_n}{a_n} = L$$

$L = 0 \Longrightarrow$ Conv. or Div. depend on b_n.

$L > 0 \Rightarrow$ Conv. or Div. depend on b_n.

Ex2- $\sum_{n=1}^{\infty} \frac{1}{3n+2} \cdot \frac{1}{n^{3/2}}$ (Div. or Conv. ?)

Sol.

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1/n}{1/3n+2} = \frac{3n+2}{n} \quad \div n$$

$$= \frac{3 + 2/n}{1} = 3 > 0 , \frac{1}{n} \text{ Div.}$$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1/n^{3/2}}{1/n^{3/2}} = 1$$

$\therefore$ Div. by L.C.

4

Ex:-

$$\sum_{n=1}^{\infty} \frac{\tan^{-1} K}{1+K^2} \quad (\text{Conv. or Div.})$$

Sol.

$$\sum_{n=1}^{\infty} \frac{\tan^{-1} K}{1+K^2} \leqslant \frac{1}{K^2}, \quad C_n = \frac{1}{K^2} \text{ Conv.}$$

$\therefore a_n$ Conv. by Comparison.

Ex:

$$\sum_{n=1}^{\infty} \frac{n^{4/3}}{8n^2+5n+1}$$

SoL. by using L.Comp. $b_n = \frac{n^{4/3}}{8n^2} = \frac{1}{8n^{2/3}}$

$$\therefore \lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1/8n^{2/3}}{n^{4/3}/8n^2+5n+1}$$

$$= \frac{8n^2+5n+1}{8n^{2/3}\cdot n^{4/3}}$$

$$= \frac{8n^2+5n+1}{8n^2} = 1 > 0$$

$\because b_n = \frac{1}{8n^{2/3}}$ Div. $(P<1)$

$\therefore \sum a_n$ Div. by L.Comp. Test.

5

(B) Test on non +ve series

$$\sum_{n=K}^{\infty} (-1)^{n+1} a_n .$$

(7) Absolute

$$\sum |a_n|$$

- تكون المتسلسلة $\sum a_n$ Conv. اذا كانت $\sum |a_n|$ Conv. أيضاً.
- تستخدم لجميع المتواليات التي تتغير اشاراتها سواء كان التغير منتظم ام مبعثر مثل $\sum \frac{\sin n}{n^2-1}$

(8) Alternative $\sum (-1)^{n+1} a_n$

- نجعل المتوالية المتذبذبة وكأنها موجبة الحدود
اي، $\sum (-1)^{n+1} a_n = \sum a_n$
- نختبر المتوالية المجردة من الاشارات بالطريقة الموجودة في A ، فاذا كانت:

Conv. $\longrightarrow$ absolately Conv.

Div. $\longrightarrow$ conditionally Conv. when Conditions below are acheived:-

١- (1) a_n = +ve terms.
(2) a_n = decreasing sequence.
(3) $\lim_{n\to\infty} a_n = 0$

وعند تحقق الشروط، تدعى العملية بـ نظرية ليبنز (Leibins theorm).
- واذا لم يتحقق اي شرط منها فانها Div.

(9) Absolute Ratio

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

- تستخدم لايجاد فترة التقارب للمتواليات ...

Ex:- Test: $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$

Sol.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n)!} \Rightarrow \rho = \lim_{n\to\infty} \frac{1/(2n+2)!}{1/(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= 0 < 1$$

$\therefore$ absolute Conv.

6

Lecturer: M.Sc Murtadha Mohsen Al-Masoudy
E-mail: Murtadha_Almasoody@mustaqbal-college.edu.iq

Ex: Test the following series.

ⓐ $\sum_{2}^{\infty}(-1)^K \frac{\ln K}{\ln K^2} \longrightarrow \sum \frac{\ln K}{\ln K^2} = \sum \frac{\ln K}{2\ln K}$
$= \sum \frac{1}{2}$ div.

Conditions

① $a_n = +ve$ terms.
② $\frac{1}{2} \geq \frac{1}{2} \geq \frac{1}{2} \geq \frac{1}{2}$
③ $\lim_{K\to\infty} \frac{\ln K}{\ln K^2} = \frac{1}{2} \neq$ zero

$\therefore$ Div.

ⓑ $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{n^3+1} \longrightarrow \sum_{1}^{\infty} \frac{n}{n^3+1}$, by using L. Comp.
$b_n = \frac{1}{n^2}$

$\therefore \lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1/n^2}{n/n^3+1}$
$= \frac{n^3+1}{n^3}$
$= 1+0 = 1$
Conv. by L. Comp.
$\therefore$ absolutly Conv.

ⓒ $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n+3} \longrightarrow \sum_{1}^{\infty} \frac{1}{n+3}$
$\lim_{n\to\infty} \frac{1/n}{1/n+3} = \frac{n+3}{n} = 1 > 0$
$b_n = \frac{1}{n} =$ Div.

Conditions
① $a_n = +ve$ terms.
② $\frac{1}{4} \geq \frac{1}{5} \geq \frac{1}{6} \geq \frac{1}{7} \geq \cdots$ decreasing.
③ $\lim_{n\to\infty} \frac{1}{n+3} = 0$

$\therefore$ Conditionally Converge.

7

Lecturer: M.Sc Murtadha Mohsen Al-Masoudy
E-mail: Murtadha_Almasoody@mustaqbal-college.edu.iq

H.W

① Determine whether the sequences are Converges.

$$\left\{\frac{n}{\ln(n+1)}\right\}, \left\{\frac{e^n}{2^n}\right\}$$

② Find the n^{th} element of sequences and tell whether the sequence Converges.

$$\left\{1, \frac{1}{1-1/2}, \frac{1}{1-2/3}, \frac{1}{1-3/4}, \cdots\right\} ; \left\{1, \frac{2}{2^2-1^2}, \frac{3}{3^2-2^2}, \frac{4}{4^2-3^2}, \cdots\right\}$$

③ Test $\left\{\cosh\left(\frac{\sin n}{n}\right) - e^{\left(\frac{\sin^2 n}{2n}\right)}\right\}$

④ Prove the sequence $\{a_n\} = \left(1+\frac{x}{n}\right)^n$ converes to limit e^x.
what about $\{b_n\} = \left(1-\frac{x}{n}\right)^{2n}$?

⑤ Test $\left\{\frac{x^{n+1}}{(n+1)!} \cdot \frac{1+\ln n}{n}\right\}$

⑥ Test $\left\{\sqrt[n]{3n+5} \cdot \left(1+\frac{1}{n}\right)^{2n}\right\}$.

⑦ Test $\left\{\frac{6^n \cdot 6^n}{n!}\right\}$; $\left\{\left[\left(1-\frac{1}{n}\right)\cdot\left(1+\frac{1}{n}\right)\right]^n\right\}$

① Does $\sum_{K=1}^{\infty} \frac{\tan^{-1} K}{1+K^2}$ have sum ?

② $\sum_{n=1}^{\infty} n^{50} \cdot e^{-n}$; $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3+n}{5+n}$; $\sum_{2}^{\infty} (-1)^n \frac{1}{(\ln n^3)}$.

③ $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+2}{3n-1}\right)^n$

④ $\sum_{1}^{\infty} \frac{\cos n\pi}{n}$.

⑤ $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\sqrt{n+1} - \sqrt{n}\right)$.

⑥ $\sum_{K=1}^{\infty} \operatorname{sech} K$.

8