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Gradients and Directional Derivatives :-

(1) If the partial derivatives of $f(x,y,z)$ are defined at $P_0(x_0,y_0,z_0)$; then the gradient of f at P_0 is the vector

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

(2) The directional derivative is defined as

$$(D_u f)_{P_0} = (\nabla f)_{P_0} \cdot u$$

where u is a unit vector

Ex:- Find the directional derivative of $f(x,y,z) = xy^2z^3$ at the point $P(3,2,1)$ in the direction towards $Q(5,3,2)$.

Solution:-

$$\overrightarrow{PQ} = (5-3)i + (3-2)j + (2-1)k = \boxed{2i + j + k}$$

$$u = \frac{\overrightarrow{PQ}}{|PQ|} = \frac{2i+j+k}{\sqrt{2^2+1^2+1^2}} = \boxed{\frac{1}{\sqrt{6}}(2i+j+k)}$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$= y^2z^3 i + 2xyz^3 j + 3xy^2z^2$$

$$(\nabla f)_p = (2^2 * 1^3) i + (2*3*2*1^3) j + (3*3*2^2*1^2) k$$

$$= \boxed{4i + 12j + 36k}$$

$$\therefore (D_u f)_p = (\nabla f)_p \cdot u$$

$$= (4i + 12j + 36k) \cdot \left(\frac{1}{\sqrt{6}}(2i + j + k)\right)$$

$$= \frac{1}{\sqrt{6}}(8 + 12 + 36) = \boxed{\frac{56}{\sqrt{6}}}$$

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Ex:- Find the directional derivative of $f(x,y,z)=x^3-xy^2-z$ at $P_o(1,1,0)$ in the direction of vector $A=2i-3j+6k$.

Solution:-

$$U=\frac{\vec{A}}{|\vec{A}|}=\frac{2i-3j+6k}{\sqrt{(2)^2+(-3)^2+(6)^2}}=\frac{2i-3j+6k}{7}$$

$$\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$$

$$=(3x^2-y^2)i+(-2xy)j+(-1)k$$

$$(\nabla f)_{P_o}=(3*1^2-1^2)i+(-2*1*1)j-k$$

$$=2i-2j-k$$

$$\therefore (D_u f)_{P_o}=(\nabla f)_{P_o}\cdot u$$

$$=(2i-2j-k)\cdot\left(\frac{2i-3j+6k}{7}\right)$$

$$=\frac{4+6-6}{7}=\boxed{\frac{4}{7}}$$

Note:-

(1) If $f(x,y,z)=$ function then

$$\overrightarrow{\text{grad}}\, f=\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$$

(2) If $\vec{F}(x,y,z)=f_1(x,y,z)i+f_2(x,y,z)j+f_3(x,y,z)k$ then

$$\text{Divergence}\,\vec{F}=\vec{\nabla}\cdot\vec{F}=\text{div}\,\vec{F}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

$$\therefore \overrightarrow{\text{Curl}}\,F=\vec{\nabla}\times\vec{F}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

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Maxima, Minima and Saddle Points :-

If $f(x,y)$ is a function of two independent variables (x,y) and the interior points (a,b) are found at $f_x = f_y = 0$. then:

1. f has a local maximum at (a,b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b)
2. f has a local minimum at (a,b) if $f_{xx} > 0$ and $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$ at (a,b)
3. f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b) where (a,b) is the critical point

Ex:- Find the extreme value of $f(x,y) = x^2 + y^2$

Solution:-

$f_x = 2x \quad , \quad f_x = 0 = 2x \Rightarrow x = 0 = a$

$f_y = 2y \quad , \quad f_y = 0 = 2y \Rightarrow y = 0 = b$

$f_{xx} = 2 \quad , \quad f_{yy} = 2 \quad , \quad f_{xy} = 0 \Rightarrow f_{xy}^2 = 0$

$f_{xx} f_{yy} - f_{xy}^2 = 2 * 2 - 0 = 4 > 0$

$\because f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

and $(a,b) = (0,0)$ is the critical point

$\therefore f(x,y) = x^2 + y^2$ has a local minimum at (a,b)

$f(0,0) = 0$

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Ex:- Find the extreme value of the function

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

Solution:-

$f_x = y - 2x - 2 \Rightarrow \boxed{f_{xx} = -2}$

$f_x = 0 \Rightarrow y - 2x - 2 = 0$ —— ①

$f_y = x - 2y - 2 \Rightarrow \boxed{f_{yy} = -2}$

$f_y = 0 \Rightarrow x - 2y - 2 = 0$ —— ②

لايجاد قيمة x و y من المعادلتين (١ و ٢) يتم حل المعادلتين آنياً

$y - 2x - 2 = 0$ —— ①
$-2y + x - 2 = 0$ —— ② $*2$

$y - 2x - 2 = 0$ —— ①
$-4y + 2x - 4 = 0$ —— ② بالجمع

$-3y - 6 = 0 \Rightarrow 3y = -6 \Rightarrow y = -2 = b$

نعوض قيمة y في المعادلة (١) لايجاد قيمة x

$-2 - 2x - 2 = 0 \Rightarrow -4 = 2x$

$x = -2 = a$

$\therefore (a,b) = (-2,-2)$

$f_{xx}f_{yy} - f_{xy}^2$

$$\boxed{\begin{array}{l} * f_{xx} = -2 \\ * f_{yy} = -2 \end{array}}$$

$-2 * -2 - (1)^2 = 3 > 0$

$\because f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$

$\therefore f(x,y)$ has a local maximum at $(-2,-2)$

$f(-2,-2) = 8$ $\boxed{8}$