# Lecture Five

# Capacitors & Inductors

## 5.1 Introduction

So far we have limited our study to resistive circuits. In this lecture, we shall introduce two new and important passive linear circuit elements: the capacitor and the inductor. Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called storage elements. We begin by introducing capacitors and describing how to combine them in series or in parallel. Then, we do the same for inductors.

## 5.2 Capacitors

A capacitor is a passive element designed to store energy in its electric field. Besides resistors, capacitors are the most common electrical components. Capacitors are used extensively in electronics, communications, computers, and power systems. For example, they are used in the tuning circuits of radio receivers and as dynamic memory elements in computer systems.

**A capacitor consists of two conducting plates separated by an insulator (or dielectric).**

In many practical applications, the plates may be aluminum foil while the dielectric may be air, ceramic, paper, or mica.

The amount of charge stored, represented by q, is directly proportional to the applied voltage **v** so that

$$q = Cv \tag{5.1}$$

where C, the constant of proportionality, is known as the capacitance of the capacitor. The unit of capacitance is the farad (**F**), in honor of the English physicist Michael Faraday (1791–1867). From **Eq. (5.1)**, we may derive the following definition.

**Capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).**

Note from **Eq. (5.1)** that 1 farad = 1 coulomb/volt.

Although the capacitance **C** of a capacitor is the ratio of the charge **q** per plate to the applied voltage **v**, it does not depend on **q** or **v**. It depends on the physical dimensions of the capacitor. The capacitance is given by

$$C = \frac{\epsilon A}{d} \qquad (5.2)$$

where **A** is the surface area of each plate, **d** is the distance between the plates, and **ε** is the permittivity of the dielectric material between the plates. Typically, capacitors have values in the **picofarad** (**pF**) to **microfarad** (**μF**) range. **Figure 5.1** shows the circuit symbols for fixed and variable capacitors.

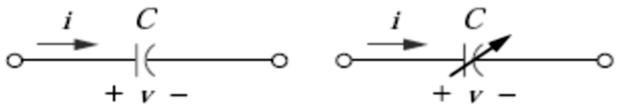



**Figure 5.1 Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.**

To obtain the current-voltage relationship of the capacitor, we take the derivative of both sides of **Eq. (5.1)**. Since

$$i = dq/dt \qquad (5.3)$$

differentiating both sides of **Eq. (5.1)** gives

$$i = C\, dv/dt \qquad (5.4)$$

The voltage-current relation of the capacitor can be obtained by integrating both sides of **Eq. (5.4)**. We get

$$v = \frac{1}{C} \int_{-\infty}^{t} i\, dt \qquad (5.5)$$

or

$$v = \frac{1}{C} \int_{t_0}^{t} i \; dt + v(t_0) \qquad (5.6)$$

where $\mathbf{v(t_0) = q(t_0)/C}$ is the voltage across the capacitor at time $\mathbf{t_0}$.

**Eq. (5.6)** shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory—a property that is often exploited.

The instantaneous power delivered to the capacitor is

$$p = vi = Cv \frac{dv}{dt} \qquad (5.7)$$

The energy stored in the capacitor is therefore

$$w = \frac{1}{2}Cv^2 \quad \text{or} \quad w = \frac{q^2}{2C} \tag{5.8}$$

**Eq. (5.8)** represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy

We should note the following important properties of a capacitor:

1. Note from **Eq. (5.4)** that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus,

**A capacitor is an open circuit to dc.**

2. The voltage on the capacitor must be continuous.

**The voltage on a capacitor cannot change abruptly.**

The capacitor resists an abrupt change in the voltage across it.

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.

4. A real, nonideal capacitor has a parallel-model leakage resistance. The leakage resistance may be as high as 100 MΩ and can be neglected for most practical applications.

**Example 5.1:**

(a) Calculate the charge stored on a 3-pF capacitor with 20 V across it.

(b) Find the energy stored in the capacitor.

**Solution:**

(a) Since **q = Cv**, $\quad \mathbf{q = 3 \times 10^{-12} \times 20 = 60\ pC}$

(b) The energy stored is $\quad w = \frac{1}{2}Cv^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600\ \text{pJ}$

**Example 5.2:**

The voltage across a 5-μF capacitor is **v(t) = 10 cos 6000t V** Calculate the current through it.

**Solution:**

By definition, the current is

$$i(t) = C\frac{dv}{dt} = 5 \times 10^{-6}\frac{d}{dt}(10\cos 6000t)$$

$$= -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t = -0.3 \sin 6000t \text{ A}$$

## 5.3 Series and Parallel Capacitors

We know from resistive circuits that series-parallel combination is a powerful tool for reducing circuits. This technique can be extended to series-parallel connections of capacitors, which are sometimes encountered. We desire to replace these capacitors by a single equivalent capacitor $\mathbf{C_{eq}}$. First we obtain the equivalent capacitor $\mathbf{C_{eq}}$ of **N** capacitors in parallel,

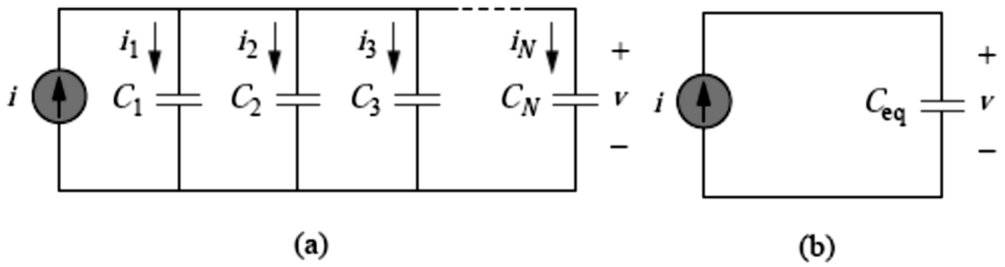



**Figure 5.2 (a) Parallel-connected N capacitors, (b) equivalent circuit for the parallel capacitors.**

$$\mathbf{C_{eq} = C_1 + C_2 + C_3 + \cdot\ \cdot\ \cdot + C_N} \qquad (5.9)$$

The equivalent capacitance of *N* parallel-connected capacitors is the sum of the individual capacitances.

We observe that capacitors in parallel combine in the same manner as resistors in series.

Now we will obtain $\mathbf{C_{eq}}$ of **N** capacitors connected in series

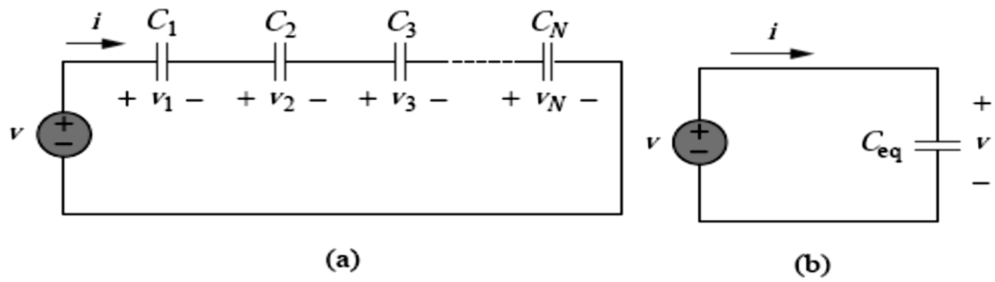



**Figure 5.3 (a) Series-connected N capacitors, (b) equivalent circuit for the series capacitor.**

Where $$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdot\cdot\cdot + \frac{1}{C_N} \qquad (5.10)$$

The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

Note that capacitors in series combine in the same manner as resistors in parallel. For **N = 2** (i.e., two capacitors in series), **Eq. (5.10)** becomes

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{Or} \quad C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \qquad (5.11)$$

**Example 5.3:**

Find the equivalent capacitance seen between terminals **a** and **b** of the circuit in **Fig. 5.4**.

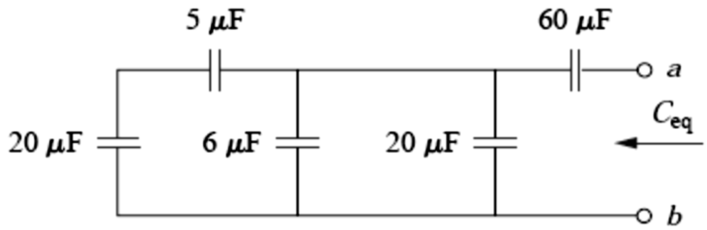



**Figure 5.4 For Example 5.3.**

**Solution:**

The 20-μF and 5-μF capacitors are in series; their equivalent capacitance is

$$\frac{20 \times 5}{20+5} = 4\ \mu F$$

This 4-μF capacitor is in parallel with the 6-μF and 20-μF capacitors; their combined capacitance is

$$4 + 6 + 20 = 30\ \mu F$$

This 30-μF capacitor is in series with the 60-μF capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{eq} = \frac{30 \times 60}{30+60} = 20\ \mu F$$

### 5.4 First Order RC Circuit

Now that we have considered the three passive elements (resistors, capacitors, and inductors), we are prepared to consider circuits that contain various combinations of two or three of the passive elements.

We carry out the analysis of ***RC*** and ***RL*** circuits by applying Kirchhoff's laws, as we did for resistive circuits. The only difference is that applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to ***RC*** and ***RL*** circuits produces differential equations, which are more difficult to solve than algebraic equations.

The differential equations resulting from analyzing *RC* and *RL* circuits are of the first order. Hence, the circuits are collectively known as ***first-order*** circuits.

A first-order circuit is characterized by a first-order differential equation.

The two types of first-order circuits and the two ways of exciting them add up to the four possible situations we will study in this chapter.

## 5.5 The Source-Free *RC* Circuit

A source-free ***RC*** circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

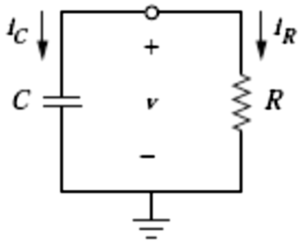



**Figure 5.5 A source-free *RC* circuit.**

Consider a series combination of a resistor and an initially charged capacitor, as shown in **Fig. 5.5**. Our objective is to determine the circuit response, which, for pedagogic reasons, we assume to be the voltage $v(t)$ across the capacitor. Since the capacitor is initially charged, we can assume that at time $t = 0$, the initial voltage is

$$v(0) = V_0 \qquad (5.12)$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV_0^2 \qquad (5.13)$$

Applying KCL at the top node of the circuit in **Fig. 5.5**,

$$i_C + i_R = 0 \qquad (5.14)$$

By definition, $i_C = C\, dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \qquad (5.15)$$

This is a ***first-order differential equation***, since only the first derivative of $v$ is involved. After solve it, the capacitor voltage is

$$v(t) = V_0 e^{-t/RC} \qquad (5.16)$$

This shows that the voltage response of the *RC* circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the *natural response* of the circuit.

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The natural response is illustrated graphically in **Fig. 5.6**. Note that at $t = 0$, we have the correct initial condition as in **Eq. (5.12)**. As $t$ increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the *time constant*, denoted by the lower case Greek letter tau, $\boldsymbol{\tau}$.

The time constant of a circuit is the time required for the response to decay by a factor of $1/e$ or 36.8 percent of its initial value.

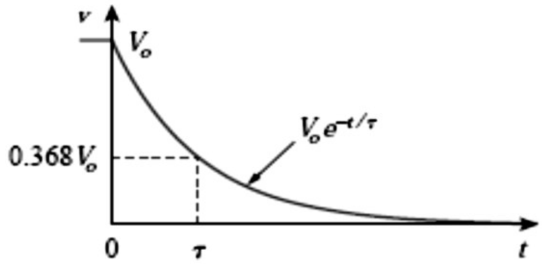



**Figure 5.6 The voltage response of the *RC* circuit.**

This implies that at $t = \tau$ , **Eq. (5.16)** becomes

$$V_0 e^{-\tau/RC} = V0e^{-1} = 0.368V0$$

or

$$\tau = RC \tag{5.17}$$

In terms of the time constant, **Eq. (5.16)** can be written as

$$v(t) = V_0 e^{-t/\tau} \tag{5.18}$$

Observe from **Eq. (5.17)** that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response. This is illustrated in **Fig. 5.7**. A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state. At any rate, whether the time constant is small or large, the circuit reaches steady state in five time constants.

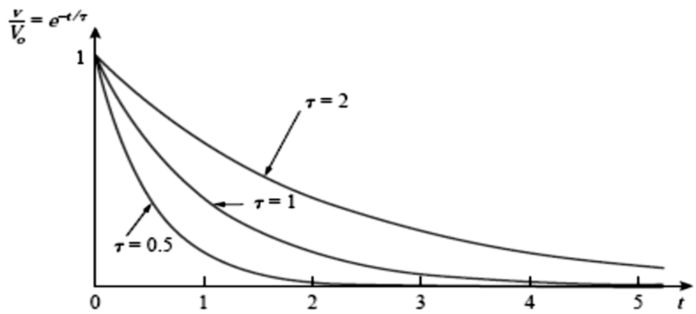



**Figure 5.7 Plot of $v/V0 = e{-}t/\tau$ for various values of the time constant.**

With the voltage *v(t)* in **Eq. (5.18)**, we can find the current *iR(t)*,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R}\, e^{-t/\tau} \qquad (5.19)$$

The power dissipated in the resistor is

$$p(t) = v.i_R = \frac{V_0^2}{R}\, e^{-2t/\tau} \qquad (5.20)$$

The energy absorbed by the resistor up to time *t* is

$$w_R(t) = \frac{1}{2} C V_0^2\, (1 - e^{-2t/\tau}),\ \ \tau = RC \qquad (5.21)$$

Notice the energy that was initially stored in the capacitor is eventually dissipated in the resistor.

In summary:

**The Key to Working with a Source - free *RC* Circuit is Finding:**

1. The initial voltage $v(0) = V0$ across the capacitor.
2. The time constant $\tau$ .

With these two items, we obtain the response as the capacitor voltage $v_C(t) = v(t) = v(0)e^{-t/\tau}$ . Once the capacitor voltage is first obtained, other variables (capacitor current $i_C$, resistor voltage $v_R$, and resistor current $i_R$) can be determined. In finding the time constant $\tau = RC$, *R* is often the **Thevenin** equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor *C* and find $R = R_{Th}$ at its terminals.

**<u>Example 5.4:</u>** The switch in the circuit in **Fig. 5.8** has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

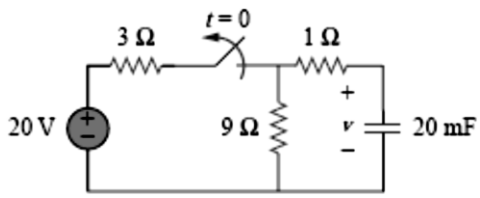



**Figure 5.8 For Example 5.4.**

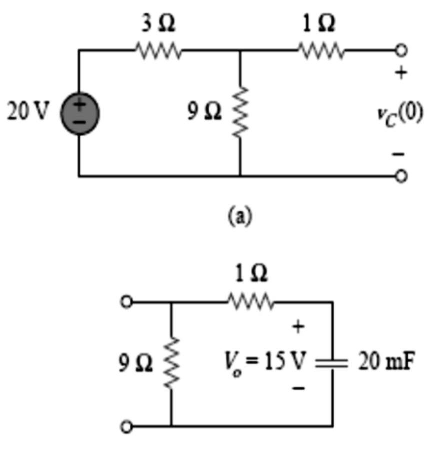
**<u>Solution:</u>**

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc, as represented in **Fig. 5.9(a)**. Using voltage division

$$v_C(t) = \frac{9}{9+3}(20) = 15\ V, t < 0$$

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15\ \text{V}$$



For $t > 0$, the switch is opened, and we have the ***RC*** circuit shown in **Fig. 5.9(b)**. [Notice that the ***RC*** circuit in **Fig. 5.9(b)** is source free; the independent source in **Fig. 5.8** is needed to provide $V_0$ or the initial energy in the capacitor.] The **1-Ω** and **9-Ω** resistors in series give

$$R_{eq} = 1 + 9 = 10\ \Omega$$

**Figure 5.9 For Example 5.4: (a) $t < 0$, (b) $t > 0$.**

The time constant is

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2\ \text{s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2}\ \text{V}$$

or $$v(t) = 15e^{-5t}\ \text{V}$$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25\ J$$

### 5.6 Step Response of an *RC* Circuit

When the dc source of an ***RC*** circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a ***step response***.

The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

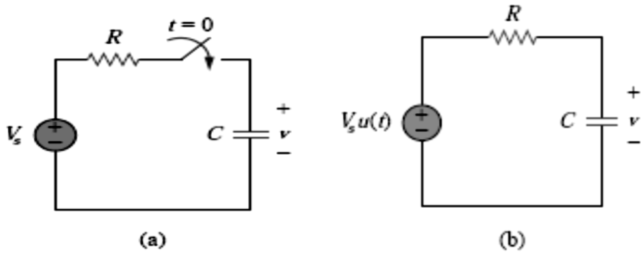



**Figure 5.10 An *RC* circuit with voltage step input.**

Consider the ***RC*** circuit in **Fig. 5.10(a)** which can be replaced by the circuit in **Fig. 5.10(b)**, where ***Vs*** is a constant, dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined.

We assume an initial voltage $V_0$ on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \qquad (5.22)$$

where *v(0−)* is the voltage across the capacitor just before switching and *v(0+)* is its voltage immediately after switching. Applying **KCL**, we have

$$C\frac{dv}{dt} + \frac{v - Vs.u(t)}{R} = 0 \tag{5.23}$$

where ***v*** is the voltage across the capacitor.

Thus,

$$v(t) = \begin{cases} V0 & , t < 0 \\ Vs + (V0 - Vs)e^{-t/\tau} & , t > 0 \end{cases} \tag{5.24}$$

This is known as the *complete response* of the *RC* circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term "complete" will become evident a little later. Assuming that *Vs* > *V*0, a plot of ***v(t)*** is shown in **Fig. 5.11**.

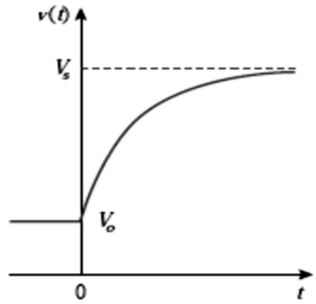



**Figure 5.11 Response of an *RC* circuit with initially charged capacitor.**

If we assume that the capacitor is uncharged initially, we set *V*0 = 0 in **Eq. (5.24)** so that

$$v(t) = \begin{cases} 0 & , t < 0 \\ Vs\,(1 - e^{-t/\tau}) & , t > 0 \end{cases} \tag{5.25}$$

Rather than going through the derivations above, there is a systematic approach—or rather, a short-cut method—for finding the step response of an ***RC*** or ***RL*** circuit. Let us reexamine **Eq. (5.24)**, which is more general than **Eq. (5.25)**. It is evident that ***v(t)*** has two components. Thus, we may write

$$v = vf + vn \tag{5.26}$$

We know that ***vn*** is the natural response of the circuit, as discussed in **Section 5.2**. Now, ***vf*** is known as the *forced* response because it is produced by the circuit when an external "force" is applied (a voltage source in this case).

The natural response or transient response is the circuit's temporary response that will die out with time.

The forced response or steady-state response is the behavior of the circuit a long time after an external excitation is applied.

The complete response of the circuit is the sum of the natural response and the forced response. Therefore, we may write **Eq. (5.24)** as

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \qquad (5.27)$$

where $v(0)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady state value. Thus, to find the step response of an ***RC*** circuit requires three things:

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant $\tau$.

Note that if the switch changes position at time $t = t0$ instead of at $t = 0$, there is a time delay in the response so that **Eq. (5.27)** becomes

$$v(t) = v(\infty) + [v(t0) - v(\infty)]e^{-(t-t0)/\tau} \qquad (5.28)$$

where $v(t0)$ is the initial value at $t = t^{+0}$ . Keep in mind that **Eq. (5.27)** or **(5.28)** applies only to step responses, that is, when the input excitation is constant.

**Example 5.5:** The switch in **Fig. 5.12** has been in position *A* for a long time. At $t = 0$, the switch moves to *B*. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.

**Solution:**

For $t < 0$, the switch is at position *A*. Since $v$ is the same as the voltage across the 5-kΩ resistor, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

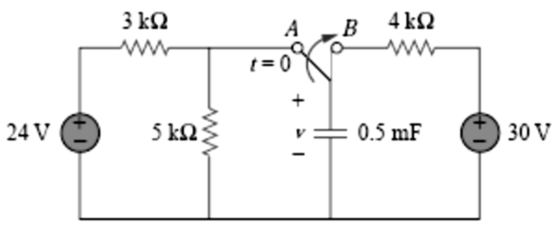



**Figure 5.12 For Example 5.5.**

$$v(0^-) = \frac{5}{5+3}(24) = 15\ V$$

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15\ \text{V}$$

For $t > 0$, the switch is in position *B*. The Thevenin resistance connected to the capacitor is $R_{Th} = 4\ \text{k}\Omega$, and the time constant is

$$\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2\ \text{s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} = 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t})\ \text{V}$$

At $t = 1$, $v(1) = 30 - 15e^{-0.5} = 20.902$ V

At $t = 4$, $v(4) = 30 - 15e^{-2} = 27.97$ V

Notice that the capacitor voltage is continuous while the resistor current is not.

## 5.7 Inductors

An inductor is a passive element designed to store energy in its magnetic field. Inductors find numerous applications in electronic and power systems. They are used in power supplies, transformers, radios, TVs, radars, and electric motors.

Any conductor of electric current has inductive properties and maybe regarded as an inductor. But in order to enhance the inductive effect, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire.

**An inductor consists of a coil of conducting wire.**

If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current. Using the passive sign convention,

$$v = L\frac{di}{dt} \qquad (5.29)$$

where L is the constant of proportionality called the inductance of the inductor. The unit of inductance is the **henry** (**H**), named in honor of the American inventor Joseph Henry (1797–1878). It is clear from Eq. (5.29) that 1 **henry** equals 1 volt-second per ampere.

**Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).**

The inductance of an inductor depends on its physical dimension and construction.

Typical practical inductors have inductance values ranging from a few **microhenrys** (**μH**), as in communication systems, to tens of henrys (**H**) as in power systems. Inductors may be fixed or variable. The core may be made of iron, steel, plastic, or air. The terms coil and choke are also used for inductors. The circuit symbols for inductors are shown in **Fig. 5.13**.

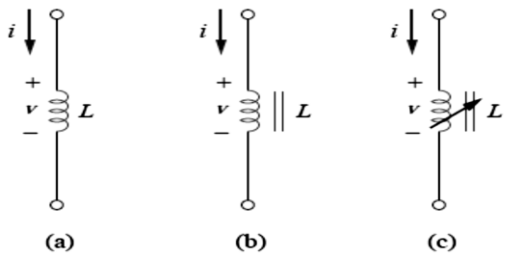



**Figure 5.13 Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core.**

The current-voltage relationship is obtained from Eq. (5.29) as

$$di = \frac{1}{L} v\, dt$$

Integrating gives $$i = \frac{1}{L}\int_{t_0}^{t} v(t)\, dt + i(t_0) \qquad (5.30)$$

where $\mathbf{i(t_0)}$ is the total current for $-\infty < t < t_0$ and $\mathbf{i(-\infty) = 0}$.

The inductor is designed to store energy in its magnetic field. The energy stored can be obtained from Eqs. (5.29) and (5.30). The power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right) i \qquad (5.31)$$

The energy stored is $$w = \frac{1}{2} L i^2 \qquad (5.32)$$

We should note the following important properties of an inductor.

1. Note from Eq. (5.29) that the voltage across an inductor is zero when the current is constant. Thus, **An inductor acts like a short circuit to dc.**

2. An important property of the inductor is its opposition to the change in current flowing through it. **The current through an inductor cannot change instantaneously.**

However, the voltage across an inductor can change abruptly.

3. Like the ideal capacitor, the ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time.

4. A practical, nonideal inductor has a significant resistive component, as shown in **Fig. 5.14**. This is due to the fact that the inductor is made of a conducting material such as copper, which has some resistance. This resistance is called the winding resistance Rw, and it appears in series with the inductance of the inductor. The presence of Rw makes it both an energy storage device and an energy dissipation device. The nonideal inductor also has a winding capacitance Cw due to the capacitive coupling between the conducting coils. Cw is very small and can be ignored in most cases, except at high frequencies. We will assume ideal inductors in this book.

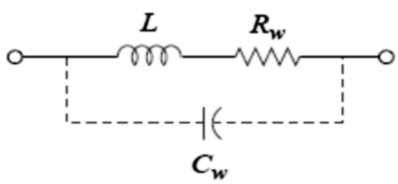



**Figure 5.14 Circuit model for a practical inductor.**

**Example 5.6:** The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

**Solution:** Since **v = Ldi/dt and L = 0.1 H,**

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1 - 5t)\,V$$

The energy stored is $w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2\,e^{-10t} = 5t^2\,e^{-10t}\,J$

**Example 5.7:** Consider the circuit in **Fig. 5.15 (a)**. Under dc conditions, find:

(a) **i**, $v_C$, and $i_L$, (b) the energy stored in the capacitor and inductor.

**Solution:** (a) Under dc conditions, we replace the capacitor with an open circuit and the inductor with a short circuit, as in **Fig. 5.15(b)**. It is evident from **Fig. 5.15(b)** that

$$i = i_L = \frac{12}{1+5} = 2\,A$$

The voltage $v_C$ is the same as the voltage across the 5-Ω resistor. Hence,

$$v_C = 5 \times i = 10\text{ V}$$

(b) The energy in the capacitor is

$$wC = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50\,J$$

and that in the inductor is

$$wL = \frac{1}{2}Li_L^2 = 12(2)(2^2) = 4\,J$$

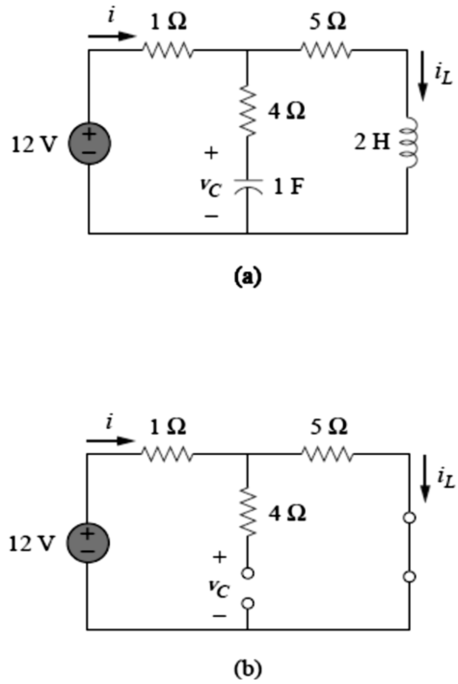



**Figure 5.15 For Example 5.7.**

### 5.8 Series and Parallel Inductors

Now that the inductor has been added to our list of passive elements, it is necessary to extend the powerful tool of series-parallel combination. We need to know how to find the equivalent inductance of a series-connected or parallel-connected set of inductors found in practical circuits.

Consider a series connection of N inductors, as shown in **Fig. 5.16(a)**, with the equivalent circuit shown in **Fig. 5.16(b)**. The inductors have the same current through them. Applying KVL to the loop,

$$v = v_1 + v_2 + v_3 + \cdots + v_N \quad (5.33)$$

Substituting $v_k = L_k\, di/dt$ results in $v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} + \cdots + L_N \frac{di}{dt}$

$$= (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt} = \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{eq}\frac{di}{dt} \quad (5.34)$$

Where $L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N$ (5.35)

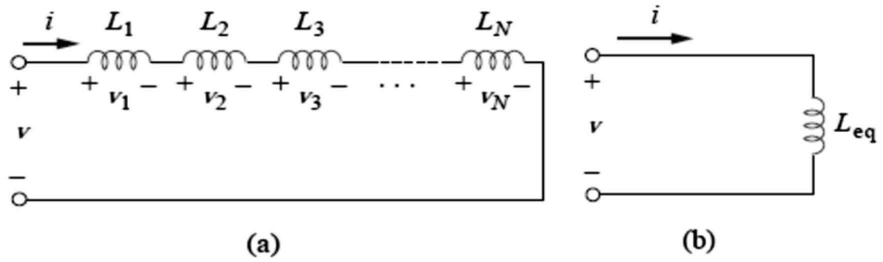



**Figure 5.16 (a) A series connection of N inductors, (b) equivalent circuit for the series inductors.**

**The equivalent inductance of series-connected inductors is the sum of the individual inductances.**

Inductors in series are combined in exactly the same way as resistors in series.

We now consider a parallel connection of N inductors, as shown in **Fig. 5.17(a)**, with the equivalent circuit in **Fig. 5.17(b)**. The inductors have the same voltage across them. Using KCL,

$$i = i_1 + i_2 + i_3 + \cdots + i_N \quad (5.36)$$

But $i_k = \frac{1}{L_k}\int_{t_0}^{t} v\,dt + i_k(t_0)$ ; hence,

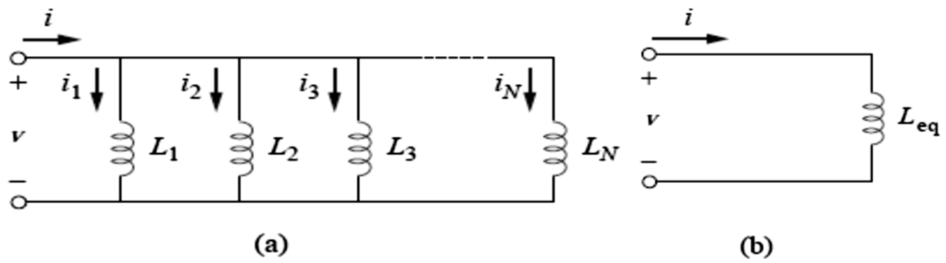



**Figure 5.17 (a) A parallel connection of N inductors, (b) equivalent circuit for the parallel inductors.**

$$i = \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0) + \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N(t_0)$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} v\,dt + i(t_0) \quad (5.37)$$

Where $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$ (5.38)

The initial current $i(t_0)$ through Leq at t = t0 is expected by KCL to be the sum of the inductor currents at $t_0$. Thus, according to **Eq. (5.37)**,

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

According to **Eq. (5.38)**,

**The equivalent inductance of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances.**

Note that the inductors in parallel are combined in the same way as resistors in parallel. For two inductors in parallel (N = 2), **Eq. (5.38)** becomes

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \text{ or } L_{eq} = \frac{L_1 L_2}{L_1 + L_2} \qquad (5.39)$$

**Example 5.8:** Find the equivalent inductance of the circuit shown in **Fig. 5.18**.

**Solution:**

The 10-H, 12-H, and 20-H inductors are in series; thus, combining them gives a 42-H inductance. This 42-H inductor is in parallel with the 7-H inductor so that they are combined, to give

$$\frac{7 \times 42}{7 + 42} = 6\ H$$

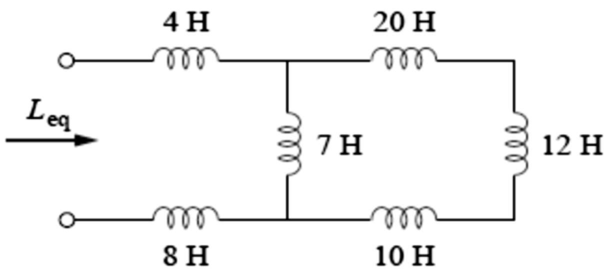



**Figure 5.18 For Example 5.8.**

This 6-H inductor is in series with the 4-H and 8-H inductors. Hence,

$$L_{eq} = 4 + 6 + 8 = 18\ H$$

## 5.9 The Source-Free RL Circuit

Consider the series connection of a resistor and an inductor, as shown in **Fig. 5.19**. Our goal is to determine the circuit response, which we will assume to be the current $i(t)$ through the inductor. We select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously. At $t = 0$, we assume that the inductor has an initial current $I_0$, or

$$i(0) = I_0 \qquad (5.40)$$

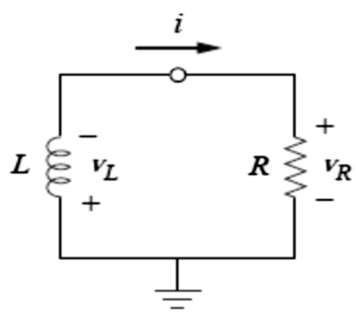



**Figure 5.19 A source-free RL circuit.**

with the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2}LI_0^2 \tag{5.41}$$

Applying KVL around the loop in **Fig. 5.19**,

$$v_L + v_R = 0 \tag{5.42}$$

But $v_L = L\frac{di}{dt}$ and $v_R = iR$. Thus, $L\frac{di}{dt} + Ri = 0$

Rearranging terms and integrating gives

$$i(t) = I_0 e^{-Rt/L} \tag{5.43}$$

This shows that the natural response of the *RL* circuit is an exponential decay of the initial current. The current response is shown in **Fig. 5.20**. The time constant for the *RL* circuit is

$$\tau = \frac{L}{R} \tag{5.44}$$

with $\tau$ again having the unit of seconds. Thus, **Eq. (5.43)** may be written as

$$i(t) = I_0 e^{-t/\tau} \tag{5.45}$$

With the current in **Eq. (5.45)**, we can find the voltage across the resistor as

$$v_R(t) = i \times R = I_0 R e^{-t/\tau} \tag{5.46}$$

The power dissipated in the resistor is

$$p = v_R \times i = I_0^2 R\, e^{-2t/\tau} \tag{5.47}$$

The energy absorbed by the resistor is

$$w_R(t) = \frac{1}{2}LI_0^2 (1 - e^{-2t/\tau}) \tag{5.48}$$

Note that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2}LI_0^2$, which is the same as $w_L(0)$, the initial energy stored in the inductor as in **Eq. (5.41)**. Again, the energy initially stored in the inductor is eventually dissipated in the resistor.

**The Key to Working with a Source - free RL Circuit is to Find :**

1. The initial current $i(0) = I_0$ through the inductor.

2. The time constant $\tau$ of the circuit.

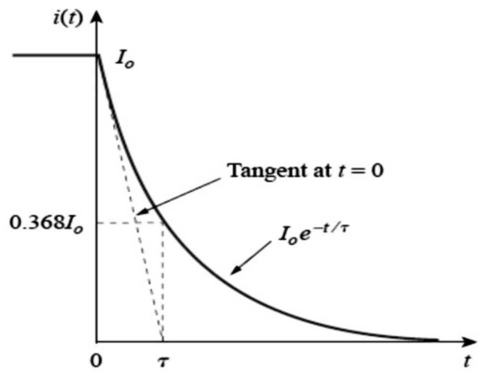



**Figure 5.20 The current Response of the RL circuit.**

Lecturer: Assist. Prof. Dr. Hamza Mohammed Ridha Al-Khafaji
E-mail: hamza.alkhafaji@uomus.edu.iq

**Example 5.9:** Assuming that $i(0) = 10$ A, calculate $i(t)$ and $ix\ (t)$ in the circuit in **Fig. 5.21.**

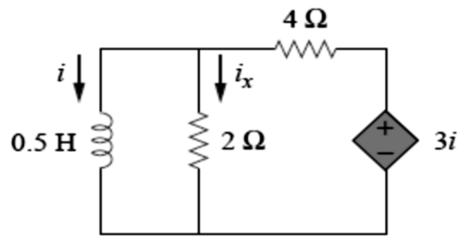



**Figure 5.21 For Example 5.9.**

**Solution:** There are two ways we can solve this problem. One way is to obtain the equivalent resistance at the inductor terminals and then use **Eq. (5.45)**. The other way is to start from scratch by using Kirchhoff's voltage law.

The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $vo = 1$ V at the inductor terminals *a*-*b*, as in **Fig. 5.22(a)**.

Applying KVL to the two loops results in

$$2(i_1 - i_2) + 1 = 0 \quad \Rightarrow \quad i_1 - i_2 = -1/2 \qquad (5.9.1)$$

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = (5/6)i_1 \qquad (5.9.2)$$

Substituting Eq. (5.9.2) into Eq. (5.9.1) gives

$$i_1 = -3 \text{ A}, \; io = -i_1 = 3 \text{ A}$$

Hence, $R_{eq} = R_{Th} = vo/io = (1/3)\Omega$

The time constant is $\tau = L/R\text{eq} = \frac{1/2}{1/3} = (3/2)\text{s}$

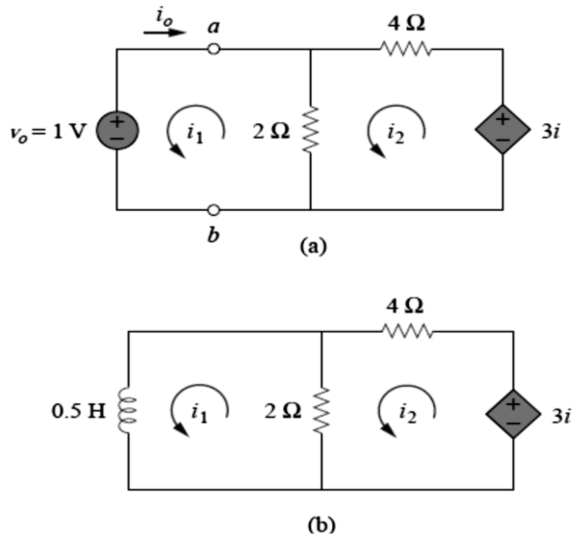



**Figure 5.22 Solving the circuit.**

Thus, the current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t} \text{ A}, \; t>0$$

The voltage across the inductor is

$$v = L\frac{di}{dt} = 0.5(10)\left(\frac{-2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t} \text{ V}$$

Since the inductor and the 2-_ resistor are in parallel,

$$ix\ (t) = v/2 = -1.667e^{-(2/3)t} \text{ A}, \; t>0$$

### 5.10 Step Response of an RL Circuit

Consider the *RL* circuit in **Fig. 5.23(a)**, which may be replaced by the circuit in **Fig. 5.23(b)**. Again, our goal is to find the inductor current ***i*** as the circuit response. Rather than apply Kirchhoff's laws. Let the response be the sum of the natural current and the forced current,

$$i = in + if \qquad (5.49)$$

We know tht the natural response is always a decaying exponential, that is,

$$in = Ae^{-t/\tau}, \qquad \tau = L/R \qquad (5.50)$$

where *A* is a constant to be determined.

The forced response is the value of the current a long time after the switch in **Fig. 5.23(a)** is closed. We know that the natural response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage $V_s$ appears across $R$. Thus, the forced response is

$$i_f = V_s/R \tag{5.51}$$

Substituting, **Eqs. (5.49)** and **(5.50)** into **Eq. (5.48)** gives

$$i = Ae^{-t/\tau} + V_s/R \tag{5.52}$$

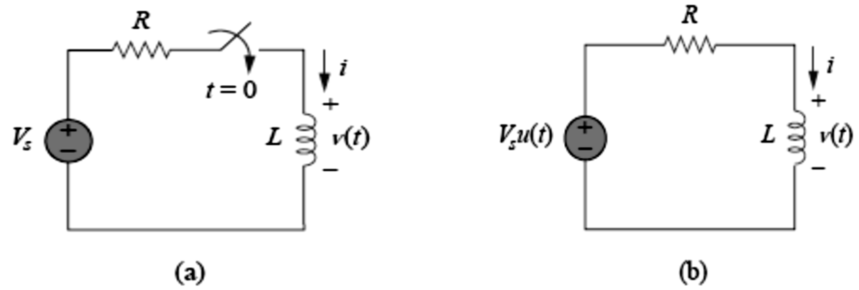



**Figure 5.23 An *RL* circuit with a step input voltage.**

We now determine the constant $A$ from the initial value of $i$. Let $I_0$ be the initial current through the inductor, which may come from a source other than $V_s$. Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \tag{5.53}$$

Thus at $t = 0$, **Eq. (5.52)** becomes

$$I_0 = A + VsR$$

From this, we obtain $A$ as

$$A = I_0 - V_s/R$$

Substituting for $A$ in **Eq. (5.52)**, we get

$$i(t) = V_s/R + (I_0 - V_s/R)e^{-t/\tau} \tag{5.54}$$

This is the complete response of the ***RL*** circuit. It is illustrated in **Fig.5.24**. The response in **Eq. (5.54)** may be written as

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \tag{5.55}$$

where $i(0)$ and $i(\infty)$ are the initial and final values of $i$. Thus, to find the step response of an $R_L$ circuit requires three things:

**1. The initial inductor current $i(0)$ at $t = 0^+$.**

**2. The final inductor current *i*(∞).**

**3. The time constant *τ*.**

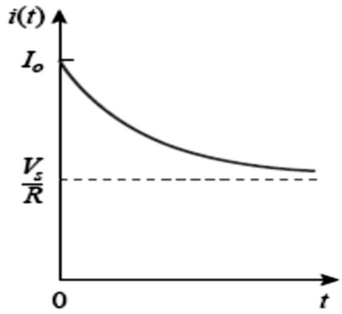



**Figure 5.24 Total response of the *RL* circuit with initial inductor current *I*0.**

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using **Eq. (5.55)**. Keep in mind that this technique applies only for step responses.

Again, if the switching takes place at time $t = t_0$ instead of $t = 0$, **Eq. (5.55)** becomes

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau} \quad (5.56)$$

If $I_0 = 0$, then

$$i(t) = \begin{cases} 0 & ,t < 0 \\ \frac{Vs}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \quad (5.57a)$$

or $$i(t) = \frac{Vs}{R}(1 - e^{-t/\tau})u(t) \quad (5.57b)$$

This is the step response of the ***RL*** circuit. The voltage across the inductor is obtained from **Eq. (5.57)** using $v = L di/dt$. We get

$$v(t) = L\frac{di}{dt} = Vs\,\frac{L}{\tau R}\,e^{-t/\tau},\ \tau = \frac{L}{R} \qquad t>0$$

or $$v(t) = Vs\,e^{-t/\tau}\,u(t) \quad (5.58)$$

**Figure 5.25** shows the step responses in **Eqs. (5.57) and (5.58).**

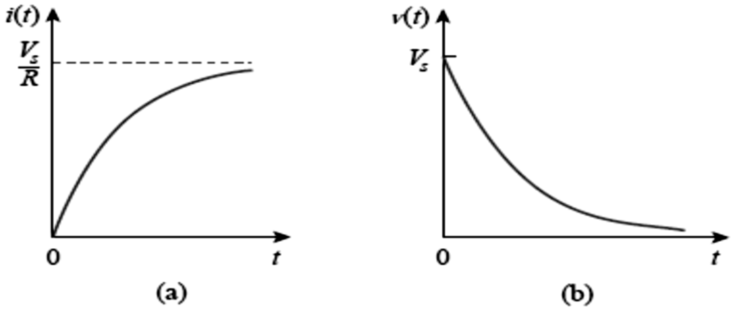



**Figure 5.25 Step responses of an *RL* circuit with no initial inductor current: (a) current response, (b) voltage response.**

**Example 5.10:** Find *i(t)* in the circuit in **Fig. 5.26** for $t > 0$. Assume that the switch has been closed for a long time.

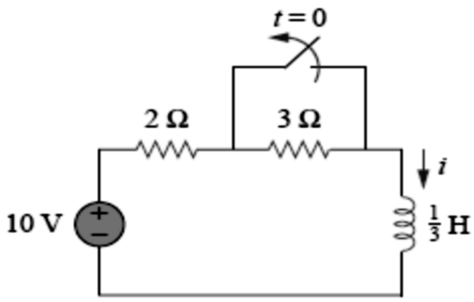



**Figure 5.26 For Example 5.10.**

**Solution:** When $t < 0$, the 3-$\Omega$ resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0-$ (i.e., just before $t = 0$) is

$$i(0^-) = 10 / 2 = 5 \text{ A}$$

Since the inductor current cannot change instantaneously,

$$i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When $t > 0$, the switch is open. The 2-$\Omega$ and 3-$\Omega$ resistors are in series, so that

$$i(\infty) = \frac{10}{2+3} = 2 \text{ A}$$

The Thevenin resistance across the inductor terminals is

$$R_{Th} = 2 + 3 = 5\ \Omega$$

For the time constant,

$$\tau = \frac{L}{R_{Th}} = \frac{1/3}{5} = 1/15 \text{ s}$$

Thus,

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} = 2 + (5-2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t>0$$