Gamma and Beta functions

The Gamma function is defined by the integral

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx \qquad ; \; n > 0$$

The Gamma function satisfies the recursive properties:

1. $\Gamma(n+1) = n\Gamma(n) \qquad \forall\, n \neq 0 \;, n \notin \mathbb{Z}^-$
2. $\Gamma(n+1) = n! \qquad n \in \mathbb{N}$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Example 1: Find 1. $\Gamma\left(\frac{3}{2}\right)$ 2. $\Gamma\left(\frac{5}{2}\right)$ 3. $\Gamma\left(-\frac{1}{2}\right)$

1. $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$
2. $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \times \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$
3. $\Gamma(n+1) = n\Gamma(n) \quad \Rightarrow \quad \Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\,\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

Example 2: Evaluate each of the following integrals

1. $\displaystyle\int_0^\infty x\sqrt{x}\; e^{-x}dx = \int_0^\infty x^{\frac{3}{2}}\, e^{-x}dx$

$$= \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2} \times \frac{1}{2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$$

2. $\int_0^\infty \frac{e^{-y^2}}{y^2} dy$

Let $x = y^2 \quad \Rightarrow \quad dx = 2ydy \quad \Rightarrow \quad dy = \frac{dx}{2\sqrt{x}}$

$$\int_0^\infty \frac{e^{-y^2}}{y^2} dy = \int_0^\infty \frac{e^{-x}}{x} \frac{dx}{2\sqrt{x}} = \frac{1}{2} \int_0^\infty x^{-\frac{3}{2}} e^{-x} dx$$

$$= \frac{1}{2} \Gamma\left(-\frac{3}{2} + 1\right) = \frac{1}{2} \Gamma\left(-\frac{1}{2}\right) = -\sqrt{\pi}$$

Beta function

The Beta function is defined by the integral

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad ; \ n > 0 \ , \qquad m > 0$$

The Beta function satisfies the recursive properties:

1. The Beta function is symmetric that is : $B(m,n) = B(n,m)$

2. $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Example 3: Evaluate 1. $B(3,4)$ 2. $B\left(\frac{1}{2}, \frac{5}{2}\right)$

$$1.\ B(3,4) = \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{2! \times 3!}{6!} = \frac{2 \times 3!}{6 \times 5 \times 4 \times 3!} = \frac{1}{60}$$

$$2.\ B\left(\frac{1}{2}, \frac{5}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{5}{2}\right)} = \frac{\sqrt{\pi} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}}{\Gamma(3)} = \frac{3\pi}{8}$$

Example 4: Evaluate each of the following integrals

$$1.\ \int_0^1 x^3(1-x)^4\, dx = B(4,5) = \frac{\Gamma(4)\Gamma(5)}{\Gamma(9)} = \frac{3! \times 4!}{8!}$$

$$= \frac{6 \times 4!}{8 \times 7 \times 6 \times 5 \times 4!} = \frac{1}{280}$$

2. $\int_0^2 \frac{x^2}{\sqrt{2-x}} dx$

Let $x = 2y \quad \Rightarrow \quad dx = 2dy$

$x = 0 \quad \Rightarrow \quad y = 0 \quad$ and $\quad x = 2 \quad \Rightarrow \quad y = 1$

$$\int_0^2 \frac{x^2}{\sqrt{2-x}} dx = \int_0^1 \frac{4y^2}{\sqrt{2-2y}} 2dy = \frac{8}{\sqrt{2}} \int_0^1 \frac{y^2}{\sqrt{1-y}} dy$$

$$= 4\sqrt{2} \int_0^1 y^2 (1-y)^{-1/2} \, dy$$

$$= 4\sqrt{2}\, B\left(3, \frac{1}{2}\right) = 4\sqrt{2} \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{64\sqrt{2}}{15}$$

Many integrals can be expressed through beta and gamma functions. One of special interest is:

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$$

Example 5 : Evaluate $\int_0^{\pi/2} \sin^9\theta \cos^5\theta \, d\theta$

$$2m - 1 = 9 \quad \Rightarrow \quad m = 5 \quad \text{and} \quad 2n - 1 = 5 \quad \Rightarrow \quad n = 3$$

$$\int_0^{\pi/2} \sin^9\theta \cos^5\theta \, d\theta = \frac{1}{2} B(5,3) = \frac{\Gamma(5)\Gamma(3)}{2\Gamma(5+3)} = \frac{4! \times 2!}{2 \times 7!} = \frac{1}{210}$$

Example 6 : Evaluate $\int_0^{\pi/2} \sin^5 x \, dx$

$$2m-1=5 \;\Rightarrow\; m=3 \quad \text{and} \quad 2n-1=0 \;\Rightarrow\; n=\frac{1}{2}$$

$$\int_0^{\pi/2} \sin^5 x \, dx = \frac{1}{2} B\left(3,\frac{1}{2}\right) = \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)} = \frac{2!\sqrt{\pi}}{2\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{8}{15}$$

Exercises

Evaluate each of the following integrals

1. $\int_0^{\infty} x^6 e^{-3x} dx$

Let $y=3x \;\Rightarrow\; x=\frac{y}{3}$ and $dx=\frac{1}{3}\,dy$

$$\int_0^{\infty} x^6 e^{-3x} dx = \int_0^{\infty} \left(\frac{y}{3}\right)^6 e^{-y} \times \frac{1}{3}\,dy = \left(\frac{1}{3}\right)^7 \int_0^{\infty} y^6 e^{-y}\,dy = \left(\frac{1}{3}\right)^7 \times 6!$$

$$= \frac{6\times5\times4\times3\times2}{3^7} = \frac{80}{243}$$

2. $\int_0^1 x^5(1-x)^6\,dx = B(4,5) = \frac{\Gamma(6)\Gamma(7)}{\Gamma(13)} = \frac{5!\times6!}{12!}$

$$= \frac{5\times4\times3\times2\times6!}{12\times11\times10\times9\times8\times7\times6!} = \frac{1}{5544}$$

3. $\int_0^{\pi/2} \cos^4 x\, dx$□

$$2m-1=4 \Rightarrow m=\frac{5}{2} \quad \text{and} \quad 2n-1=0 \Rightarrow n=\frac{1}{2}$$

$$\int_0^{\pi/2} \cos^4 x\, dx = \frac{1}{2} B\left(\frac{5}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} = \frac{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\times\sqrt{\pi}}{2\times 2} = \frac{3\pi}{16}$$□

4. $\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx$ □

$$2m-1=4 \Rightarrow m=\frac{5}{2} \quad \text{and} \quad 2n-1=5 \Rightarrow n=3$$

$$\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx = \frac{1}{2} B\left(\frac{5}{2},3\right) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{2\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\times 3\times 2}{2\times\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}} = \frac{8}{105}$$