عنوان المحاضرة: **Detreminants, Properties, Cramer's Rule, App of Det.** المحددات، الخصائص، قاعدة كرامر، تطبيقات المحددات

Determinants :- المحددات

Square array of numbers or symbles and closed by two vertical bars.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = |a_{ij}|$$

a_{ij} = individual elements.
i = row
j = column
n = order of determinant
D = (always be) value of determinant.

Miner a_{ij} :-

Designated as M_{ij} of order $n-1$

$$a_{ij} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & \widehat{a_{23}} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \Rightarrow M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

ويعني المصفوفة الثانية يتم الحذف تسمى Miner.

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Cofactor of A_{ij} :-

Designated by A_{ij} is defined by :-

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Laplace Expansion Formula :-

$$Det A = \sum_{j=1}^{n} a_{ij} A_{ij} \quad \longleftarrow \; (i = \text{constant}, \text{ summing along row } i)$$

$$Det A = \sum_{i=1}^{n} a_{ij} A_{ij} \quad \longleftarrow \; (\text{summing along column } j)$$

Example

Find the Determine of $D = \begin{vmatrix} 3 & 0 & -2 \\ 1 & 4 & 1 \\ 3 & -2 & 5 \end{vmatrix}$

Solution

- Lets do it about first column, which means $j = 1 = \text{constant}$

$$\therefore D = (3)(-1)^{1+1} \begin{vmatrix} 4 & 1 \\ -2 & 5 \end{vmatrix} + (1)(-1)^{1+2} \begin{vmatrix} 0 & -2 \\ -2 & 5 \end{vmatrix} + (3)(-1)^{1+3} \begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix}$$

$$\therefore D = 94$$

- Lets expansion it about 1st row, which means $i = 1 = \text{constant}$

$$\therefore D = (3)(-1)^{1+1} \begin{vmatrix} 4 & 1 \\ -2 & 5 \end{vmatrix} + (0)(-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} + (-2)(-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix}$$

$$= 94 \quad \text{ok}$$

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Matrices - المصفوفات

A rectangular array of numbers or symbols with each element being distinct and seperate (m rows, n columns) is called an m×n matrix when certain laws of Combination, yet to be specified, are laid down.

Example :

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad \text{where} \quad i = 1, 2, \cdots, m$$

Solution

$$y_1 = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n$$

$$\vdots$$

$$y_m = a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n$$

This can be written as a matrix

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$[Y] = [A] \qquad\qquad [X]$$

$$[Y] = [A] [X]$$

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Special Matrices :-

① **Square matrix :-**

$$[A] = \begin{bmatrix} a_{11} & -- & a_{1n} \\ a_{n1} & -- & a_{nn} \end{bmatrix} \implies (m=n)$$

② **Diagonal matrix :-**

$$[A] = \begin{bmatrix} a_{11} & 0 & -- & 0 \\ 0 & a_{22} & & 0 \\ \vdots & \vdots & a_{33} & a_{nn} \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

③ **Unit matrix :-** It may called by Identity matrix

$$[I] = \begin{bmatrix} 1 & 0 & 0 & \sim & 0 \\ 0 & 1 & 0 & -- & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

④ **Upper triangular matrix :-**

$$[U] = \begin{bmatrix} U_{11} & U_{12} & U_{13} & --- & U_{1n} \\ 0 & U_{22} & U_{23} & -- & U_{2n} \\ 0 & 0 & U_{33} & -- & U_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & -- & U_{nn} \end{bmatrix}$$

⑤ **Upper Unit triangular matrix :-**

$$[I] = [U] \quad \text{with diagonal elements equal to } 1.$$

⑥ <u>Lower triangular matrix</u> :-

$[L]$ = all elements above the diagonal are <u>zero</u>

⑦ <u>Inverse of Square matrix $[A]$</u> :-
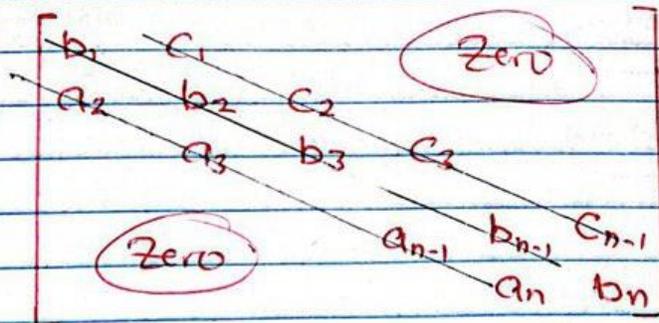Defined by:-

(Identity or unit matrix)

$$[A]^{-1} [A] = [I] \qquad a^{-1} a = \frac{a}{a} = 1$$

where,

$[A]^{-1}$ is called A inverse

⑧ <u>Tridiagonal matrix</u> :- مصفوفة قطرية

⑨ <u>Transpose of a matrix</u> :-
A matrix form when columns & rows are interchange, designated as A^T or $[A]^T$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix} \Rightarrow [A]^T = \begin{bmatrix} a_{11} & a_{21} & --- & a_{m1} \\ a_{12} & a_{22} & --- & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & --- & a_{mn} \end{bmatrix}$$

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(10) Elementry matrix operations :-

a- Two matrices are equal

$$[A] = [B] \quad \text{if} \quad a_{ij} = b_{ij} \quad , \begin{cases} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, n \end{cases}$$

b- Matrix addition :-

$$[A] + [B] = [C] \quad \text{if} \quad c_{ij} = a_{ij} + b_{ij}$$

c- Matrix multiplication :-

$$[C] = [A][B] \quad \text{if} \quad c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

$$\begin{cases} i = 1, 2, \cdots, m \\ k = 1, 2, \cdots, q \end{cases}$$

Example :-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} \overbrace{(a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31})}^{C_{11}} & \overbrace{\begin{array}{c}(a_{11}b_{12} + a_{12}b_{22} \\ + a_{13}b_{32})\end{array}}^{C_{12}} \\ \underbrace{(a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31})}_{C_{21}} & \underbrace{\begin{array}{c}a_{21}b_{12} + a_{22}b_{22} \\ + a_{23}b_{32}\end{array}}_{C_{22}} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Matrix properties :-

1- Matrix multiplication is associative

$$[A]\{[B][C]\} = \{[A][B]\}[C]$$

2- Not commutative

$$[A][B] \neq [B][A]$$

But there is exception $\Rightarrow [I][A] = [A][I] = [A]$

3- Is distributive $\Rightarrow [A]\{[B] + [C]\} = [A][B] + [A][C]$

Properties of Determinant :-

a- $\text{Det}[I] = 1$

b- $\text{Det}[A] = \text{Det}[A]^{-1}$

c- $(\text{Det}[A])^{-1} = \dfrac{1}{\text{Det}[A]}$

d- If $[A]$ and $[B]$ are square matrix, then

$$\text{Det}([A][B]) = \text{Det}[A] \cdot \text{Det}[B]$$

e- If $[A]$ has size $n \times n$ and K is a constant, then,

$$\text{Det}(K[A]) = K^n \cdot \text{Det}[A]$$

f- If to the element of any row (or column) are added K times the corresponding elements of any other row (or column), the determinant is un-changed

Example:

Find determinant $D = \begin{vmatrix} 3 & 1 & 3 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{vmatrix} = -8$

And, with property "f":

$$D = \begin{vmatrix} 3 & 1 & (3-(2\times 1)) \\ 0 & 2 & (4-(2\times 2)) \\ 1 & 2 & (3-(2\times 2)) \end{vmatrix} = -8$$

g- If any two rows (or columns) of $[A]$ are inter-changed, yielding a new matrix $[B]$, then

$$\text{Det}[B] = -\text{Det}[A]$$

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h- If $[A]$ is triangular, then $\text{Det}[A]$ is the product of the diagonal elements.

$$\text{Det}[A] = |A| = a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn}$$

__Example__

Let $\quad A = \begin{bmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{bmatrix}$

Find the $\text{Det}[A]$ with several different methods.

__Solution__

① By using "Laplace Expansion formula":-

$$\text{Det } A = \sum_{j=1}^{} a_{ij} A_{ij} \quad ; \quad A_{ij} = (-1)^{i+j} M_{ij}$$

a- Using row #1

$$\text{Det } A = |A| = (0) \begin{vmatrix} 3 & 5 \\ 0 & -4 \end{vmatrix} - (2) \begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix}$$

$$= 0 - (2)[-16 - 10] + (-1)[0 - 6]$$

$$|A| = \boxed{58}$$

b- Using row #2

$$\text{Det } A = |A| = -(4) \begin{vmatrix} 2 & -1 \\ 0 & -4 \end{vmatrix} + (3) \begin{vmatrix} 0 & -1 \\ 2 & -4 \end{vmatrix} - (5) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -(4)(-8) + (3)(2) - (5)(-4)$$

$$|A| = \boxed{58}$$

c- Using property (g), then (f) to get "upper triangular matrix"

$$|A| = \begin{vmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & -4 \\ 4 & 3 & 5 \\ 0 & 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 & -4 \\ 0 & 3 & 13 \\ 0 & 2 & -4 \end{vmatrix} \leftarrow \{a_{2j} - 2(a_{1j})\}$$

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$$= - \begin{vmatrix} 2 & 0 & -4 \\ 0 & 3 & 13 \\ 0 & 0 & -\frac{29}{3} \end{vmatrix} \longleftarrow \left\{ a_{3j} - \frac{2}{3}(a_{2j}) \right\}$$

Now we have "Upper triangular matrix"

$$= -(2)(3)\left(-\frac{29}{3}\right)$$

$$= \boxed{58}$$

i- IF all the elements of any row or column are zero, then Det A = 0

j- IF any one row (or column) a of A is separated as $a = b + c$, then

$$Det \, A|_a = Det \, A|_b + Det \, A|_c$$

Ex||

$$\begin{vmatrix} 6+2 & -3+1 & 5+4 \\ 3 & 0 & 2 \\ 1 & -6 & 7 \end{vmatrix} = \begin{vmatrix} 6 & -3 & 5 \\ 3 & 0 & 2 \\ 1 & -6 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 6 & 7 \end{vmatrix}$$

k- The determinant of A & its transpose are equal

$$Det \, A^T = Det \, A$$

l- In general $Det(A+B) \neq Det \, A + Det \, B$

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Cramer's Rule :-

{The name is coming after
Gabriel Cramer (1704-1752)}

$$X_i = \frac{\text{Det } A_i}{\text{Det } A}$$; $[A][X] = [B]$

where;

Det A = Determinant of matrix $[A]$.

Det A_i = Determinant of matrix $[A_i]$, where $[A_i]$ is matrix $[A]$ with its i-th column replaced by column matrix $[B]$.

Ex)

$$3X_1 - 3X_2 + 5X_3 = 4$$
$$X_1 + 2X_2 - 6X_3 = 3$$
$$2X_1 - X_2 + 3X_3 = 1$$

Find $X_1, X_2, \& X_3$

Solution

In the matrix form, we have:

$$\begin{bmatrix} 3 & -3 & 5 \\ 1 & 2 & -6 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

   $[A]$ $[X] = [B]$

We can find Det $A = 20$

$$\text{Det } A_1 = \begin{vmatrix} 4 & -3 & 5 \\ 3 & 2 & -6 \\ 1 & -1 & 3 \end{vmatrix} = 20$$

Similarly,

$$\text{Det } A_2 = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 3 & -6 \\ 2 & 1 & 3 \end{vmatrix} = -40$$

Det $A_3 = -20$

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Then,

$$x_1 = \frac{20}{20} = 1 \quad , \quad x_2 = \frac{-40}{20} = -2 \quad , \quad x_3 = \frac{-20}{20} = -1$$

Application of the Determinant :-

* First we need to define the **adjugate** of (A) to be the transpose of the Cofactor matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{then,}} adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A_{ij}^T$$

(Cofactor Transpose)

Inverse Matrix :-

$$A^{-1} = \frac{adj(A)}{Det(A)} \quad , \quad Det(A) \neq 0$$

EX) find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad , \quad \text{using the adjugate theorm.}$$

Solution

× First, we will find the determinant of the matrix

$$Det\,A = (1)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 1 & 6 \end{vmatrix} + 3\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= -2 - 2(3-1) + 3(6) = -2 - 4 + 18$$

Det A = 12

Now, we need to Find adj(A). To do so, 1st we will find the Cofactor matrix of (A). This is given by

$$Cof(A) = \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix} = A_{ij}$$

$$\therefore adj(A) = A_{ij}^{T} = \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{12}\begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

To verify our answer For (A⁻¹). Compute the Product A·A⁻¹ &/or A⁻¹·A & make sure each product is equal to I (identity matrix)

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

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H.W

Find the inverse of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$$

Using the "adjugate theorem"

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