Special Functions

Exponential and Logarithm functions:

Exponential functions: **If *a* is a positive number and *x* is any number, we define the exponential function as:**

$y = a^x$ **with domain:** $-\infty < x < \infty$

Range: $y > 0$

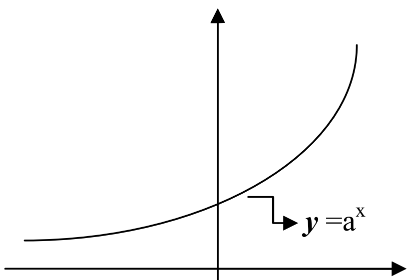
The properties of the exponential functions are:

1. If $a > 0 \Leftrightarrow a^x > 0$.
2. $a^x . a^y = a^{x+y}$
3. $a_x / a_y = a^{x-y}$
4. $(a^x)^y = a^{xy}$
5. $(a . b)^x = a^x . b^x$.
6. $a^{xly} = \sqrt[y]{a^x} = (a^{\frac{1}{y}})^x$
7. $a^{-x} = \frac{1}{a^x}$ & $a^x = 1/a^{-x}$.
8. $a^x = a^y \Leftrightarrow x = y$.
9. $a^0 = 1$,

$a^{\infty} = \infty$, $a^{-\infty} = 0$, *where* $a > 1$

$a^{\infty} = 0$, $a^{-\infty} = \infty$, *where* $a < 1$

The graph of the exponential function $y = a^x$ **is :**



Logarithm function : **If *a* is any positive number other than *1* , then the logarithm of *x* to the base *a* denoted by :**

$y = \log_a x$ *where* $x > 0$

At $a = e = 2.7182828\ldots$ **, we get the natural logarithm and denoted by :**

$y = \ln x$

Let $x , y > 0$ **then the properties of logarithm functions are :**

1. $y = a^x \Leftrightarrow x = \log_a y$ *and* $y = e^x \Leftrightarrow x = \ln y$.
2. $\log_e x = \ln x$.
3. $\log_a x = \ln x / \ln a$.
4. $\ln (x.y) = \ln x + \ln y$.
5. $\ln (x / y) = \ln x - \ln y$.
6. $\ln x^n = n. \ln x$.
7. $\ln e = \log_a a = 1$ *and* $\ln 1 = \log_a 1 = 0$.
8. $a^x = e^{x \ln a}$
9. $e^{\ln x} = x$.

The graph of the function $y = ln\ x$ is :

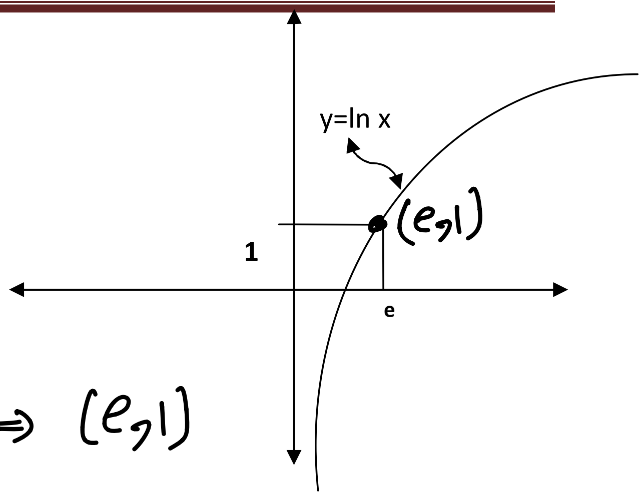
$\boldsymbol{y = ln\ x \Leftrightarrow x = e^y,\ e = 2.7}$

$\boldsymbol{Domain}: (\mathbf{0}, \infty)$

$\boldsymbol{Range}: \boldsymbol{R}$

if $x = e \Rightarrow f(x) = y = \ln x$

$f(e) = \ln e = 1 \Rightarrow (e,1)$

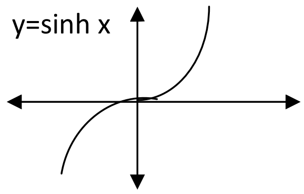


Hyperbolic Functions:

The Hyperbolic Functions are certain combinations of the exponential functions

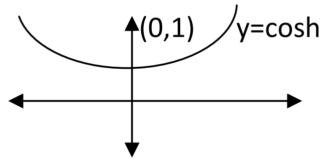
e^x *and* e^{-x} *they are*:

(i)Hyperbolic Sine (Sinh): $y = \sinh x = \frac{e^x - e^{-x}}{2}$



Domain: R, *Range*: R

(ii)Hyperbolic Cosin (cosh): $y = \cosh x = \frac{e^x + e^{-x}}{2}$

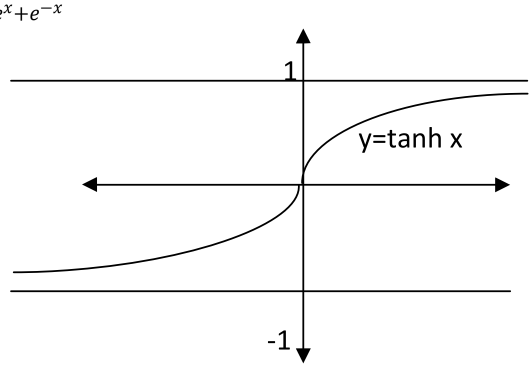


Domain: R, *Range*: $(1, \infty)$

(iii)Hyperbolic tangent (tanh): $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Domain: R

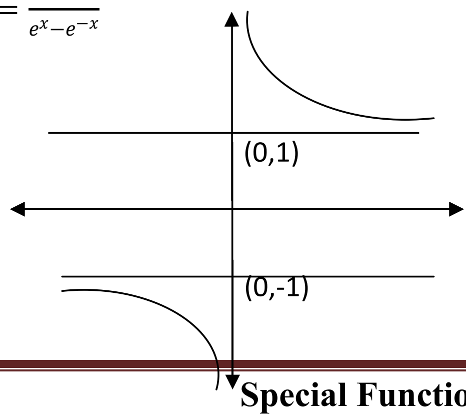
Range: $(-1,1)$



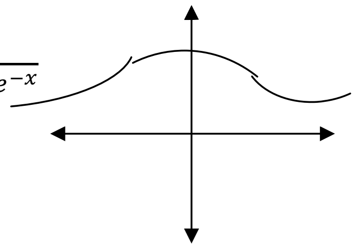
(iv)Hyperbolic cotangent (coth): $y = \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Domain: $R - \{0\}$

Range: $\{y: y < -1 \ \ or \ \ y > 1$



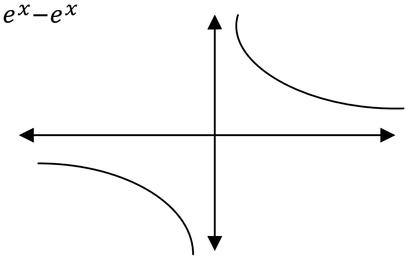
(v)Hyperbolic Secant (Sech): $y = \text{sech}\, x = \frac{2}{e^x+e^{-x}}$



Domain: R

Range: $(0,1)$

(vi)Hyperbolic cosecant (Csch): $y = \text{csch}\, x = \frac{2}{e^x-e^x}$



Domain: $R - \{0\}$

Range: $R - \{0\}$

Relationships among
Hyperbolic Function

1 - $\cosh^2 x - \sinh^2 x = 1$
2 - $\text{sech}^2 x + \tanh^2 x = 1$
3 - $\coth^2 x - \text{csch}^2 x = 1$

Functions of negative arguments

1 - $Sinh(-x) = -Sinh x$

2 - $Cosh(-x) = Cosh x$

3 - $\tanh(-x) = -\tanh x$

4 - $Coth(-x) = -Coth x$

5 - $Sech(-x) = Sech x$

6 - $Csch(-x) = -Csh x$

Addition Formula:

1 - $Sinh(x \pm y) = Sinh x\, Cosh y \pm Cosh x\, Sinh y$

2 - $Cosh(x \pm y) = Cosh x\, Cosh y \pm Sinh x\, Sinh y$

Double angle formula:

1 – $\sinh 2x = 2 \sinh x \cosh x$
2 - $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 1 + 2\sinh^2 x$
$= 2\cosh^2 x - 1$

Ex: Let $\tanh u = -\frac{7}{25}$, *determine the values of the remaining five hyperbolic functions.*

Sol:

$\coth u = \frac{1}{\tanh u} = -\frac{25}{7}$,

$\tanh^2 u + \operatorname{sech}^2 u = 1 \;\Rightarrow \frac{49}{625} + \operatorname{sech}^2 u = 1 \Rightarrow \operatorname{sech} u = \frac{24}{25}$

$\cosh u = \frac{1}{\operatorname{sech} u} = \frac{25}{24}$,

since $\tanh u = \frac{\sinh u}{\cosh u} \;\Rightarrow -\frac{7}{25} = \frac{\sinh u}{\frac{25}{24}} \;\Rightarrow \sinh u = -\frac{7}{24}$

$\operatorname{csch} u = \frac{1}{\sinh u} = -\frac{24}{7}$.

Ex: Rewrite the following expressions in terms in terms of exponentials Write the final result as simply as you can:

$a) 2\cosh(\ln x)$, $\quad b) \tanh(\ln x)$

$c) \cosh 5x + \sinh 5x$, $\quad d) (\sinh x + \cosh x)^4$.

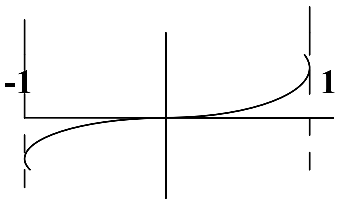
Sol:

$a)\ 2\cosh(\ln x) = 2.\frac{e^{\ln x}+e^{-\ln x}}{2} = x + \frac{1}{x}$

$b)\ \tanh(\ln x) = \frac{e^{\ln x}-e^{-\ln x}}{e^{\ln x}+e^{-\ln x}} = \frac{x-\frac{1}{x}}{x+\frac{1}{x}} = \frac{x^2-1}{x^2+1}$

$c)\ \cosh 5x + \sinh 5x = \frac{e^{5x}+e^{-5x}}{2} + \frac{e^{5x}-e^{-5x}}{2} = e^{5x}$

$d)\ (\sinh x + \cosh x)^4 = \left(\frac{e^{x}-e^{-x}}{2} + \frac{e^{x}+e^{-x}}{2}\right)^4 = e^{4x}$.



Inverse Trigonometric Function

1 – Inverse Sine ($\sin^{-1}$)

$y = \sin^{-1} x = arc\ sin \Leftrightarrow x = \sin y$

$\sin y = \sin \sin^{-1} x \rightarrow \sin y = x$

$45° = \sin^{-1}\frac{1}{\sqrt{2}}$, $\quad \frac{\pi}{2} = \sin^{-1} 1 \quad D_x: -1 \le x \le 1$

$R_y: -90 \le y \le 90$

$\sin 45° = \sin \sin^{-1}\frac{1}{\sqrt{2}}$

$2 - Inverse\ cosine \quad (\cos^{-1} x)$

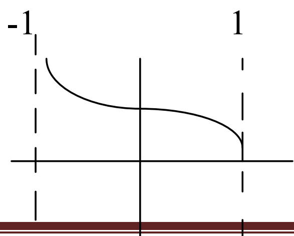
$y = \cos^{-1} x = arc\ cos \Leftrightarrow x = \cos y$

$30° = \cos^{-1}\frac{\sqrt{3}}{2}$, $\quad \frac{\sqrt{3}}{2} = \cos 30°$

$D_x: -1 \le x \le 1$, $\quad R_y: 0 \le y \le \pi$

$\cos 30 = \frac{\sqrt{3}}{2}$ (نأخذ $\cos^{-1}$ للطرفين)

$\Rightarrow \cos^{-1}\cos 30 = \cos^{-1}\frac{\sqrt{3}}{2} \Rightarrow 30° = \cos^{-1}\frac{\sqrt{3}}{2}$

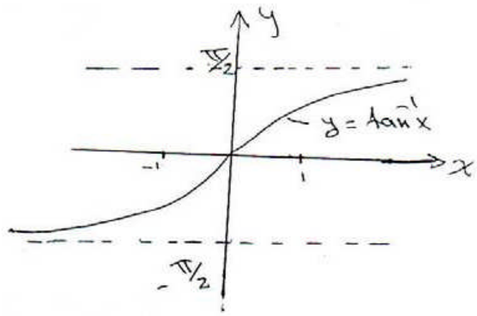


3 − *Inverse tangent* $(\tan^{-1} x)$

$y = \tan^{-1} x = arc\ tan \Leftrightarrow x = \tan y$

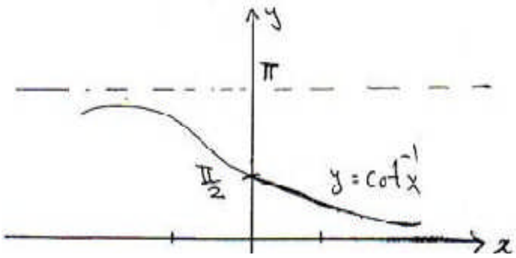
$45° = \tan^{-1} 1$, $1 = \tan 45°$

$D_x: R,\qquad R_y: -\frac{\pi}{2} < y < \frac{\pi}{2}$



4 – Inverse of cotangent $\cot^{-1} x$

$y = Cot^{-1}x = arc\ Cot\,\chi \Leftrightarrow \chi = Cot\,y$



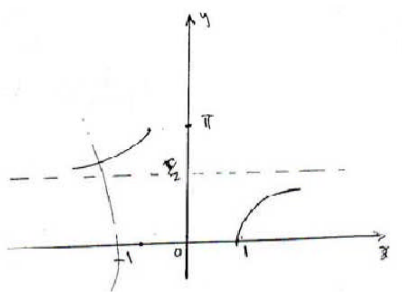
Domain: R

Range: (0 , π)

5 – Inverse secant (Sec -1)

$y = Sec^{-1}\chi = arc\ Sec\chi \rightarrow \chi = Sec\,y$

Domain: $(-\infty, -1] \cup [1, +\infty)$

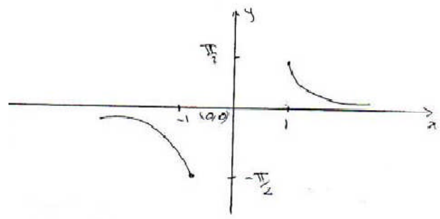


$Range: \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

6 − Inverse Csc $(\mathrm{Csc}^{-1} x)$

$y = Csc^{-1}\chi = arc\ Csc\chi \rightarrow \chi = Csc\,y$

Domain: $(-\infty, -1] \cup [1, +\infty)$



$Range: \left(-\frac{\pi}{2},0\right] \cup [0,\frac{\pi}{2})$

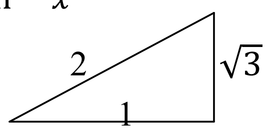
The following are some properties of the inverse trigonometric functions:

1) $\sin^{-1}(-x) = -\sin^{-1} x$
2) $\cos^{-1}(-x) = \pi - \cos^{-1} x$
3) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$
4) $\tan^{-1}(-x) = -\tan^{-1} x$
5) $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$, 6) $\sec^{-1} x = \cos^{-1} \frac{1}{x}$
7) $\csc^{-1} x = \sin^{-1} \frac{1}{x}$, 8) $\sec^{-1}(-x) = \pi - \sec^{-1} x$

and noted that $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x \neq \sin^{-1} x$

Ex: Given that $\alpha = \sin^{-1} \frac{\sqrt{3}}{2}$, *find:*

$\csc\alpha, \cos\alpha, \sec\alpha, \tan\alpha$, *and* $\cot\alpha$



Sol:

$\alpha = \sin^{-1} \frac{\sqrt{3}}{2} \Rightarrow \sin\alpha = \frac{\sqrt{3}}{2} = \frac{x}{y} \Rightarrow r = \sqrt{4-3} = 1$

$\csc\alpha = \frac{2}{\sqrt{3}}, \cos\alpha = \frac{1}{2}, \sec\alpha = 2, \tan\alpha = \sqrt{3}, and\ \cot\alpha = \frac{1}{\sqrt{3}}$

Ex: Evaluate the following expressions:

$a)\ \sec(\cos^{-1}\frac{1}{2})$, $b)\ \sin^{-1} 1 - \sin^{-1}(-1)$, $c)\ \cos^{-1}(-\sin\frac{\pi}{6})$

Sol:

$a)\ \sec(\cos^{-1}\frac{1}{2}) = \sec\frac{\pi}{3} = 2$

$b)\ \sin^{-1} 1 - \sin^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

$c)\ \cos^{-1}(-\sin\frac{\pi}{6}) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2}{3}\pi$

Ex: Prove that:

$a)\ \sec^{-1} x = \cos^{-1}\frac{1}{x}$, $b)\ \sin^{-1}(-x) = -\sin^{-1} x$

Sol:

$a)$ *Let* $y = \sec^{-1} x \Rightarrow x = \sec y \Rightarrow x = \frac{1}{\cos y} \Rightarrow y = \cos^{-1}\frac{1}{x}$

$\Rightarrow \sec^{-1} x = \cos^{-1}\frac{1}{x}$

$b)$ *let* $y = -\sin^{-1} x \Rightarrow x = \sin(-y) \Rightarrow x = -\sin y$

$\Rightarrow y = \sin^{-1}(-x) \Rightarrow \sin^{-1}(-x) = -\sin^{-1} x$