# Vectors

vectors in plane

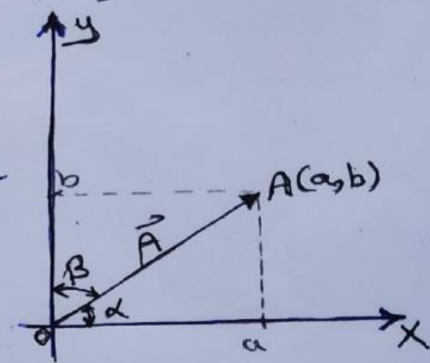
* $\vec{A} = \vec{OA} = ai + bj$

where

$i, j$ :- are the fundamental unit vectors

* Length of $\vec{A} = |A| = \sqrt{a^2 + b^2}$

* Unit vector $= \vec{U_A} = \dfrac{\vec{A}}{|A|} = \dfrac{ai + bj}{|A|}$




$$\vec{U_A} = \frac{a}{|A|} i + \frac{b}{|A|} j \quad \text{——} \; ①$$

but, $\cos\alpha = \sin\beta = \dfrac{a}{|A|}$ —— ②

$\cos\beta = \sin\alpha = \dfrac{b}{|A|}$ —— ③

Sub. eq (2) & eq (3) in eq (1) give:-

* $\vec{U_A} = \cos\alpha\, i + \cos\beta\, j$

or $\vec{U_A} = \cos\alpha\, i + \sin\alpha\, j$

* $\vec{A} = \vec{U_A} \cdot |A| = (\cos\alpha\, i + \sin\alpha\, j) \cdot |A|$

where

$a, b$ :- Direction numbers

$\alpha, \beta$ :- Direction angles

$\cos\alpha, \cos\beta$ :- Direction cosines

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**EX.(1):-** Find a vector in plane ($R^2$) of length (7 units) which makes angle ($35°$) with X-axis?

**Answer:-**

Since $|A| = 7$ & $\alpha = 35°$

$$\vec{A} = |A| (\cos\alpha \, i + \sin\alpha \, j)$$

$$\therefore \vec{A} = 7 * (\cos 35° \, i + \sin 35° \, j)$$

$$= \boxed{5.7 i + 4 j}$$

**EX(2):-** Find the angle between the vector $\vec{A} = 2i + 3j$ and the x-axis?

**Answer:-**

$$|A| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$$

$$\cos\alpha = \frac{a}{|A|} = \frac{2}{\sqrt{13}} \Rightarrow \alpha = \cos^{-1}\left(\frac{2}{\sqrt{13}}\right)$$

$$\boxed{\alpha = 56.3°}$$

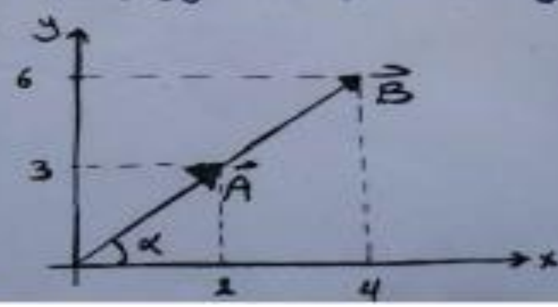
**Note:-** Two vectors are parallel if either is proportional to an other

i.e $\vec{A} \parallel \vec{B} \Leftrightarrow \vec{B} = t\vec{A}$ $\quad (\vec{U}_A = \vec{U}_B)$

where $t$ is a scalar quantity

**EX(3):-** The two vectors $\vec{A} = 2i + 3j$ and $\vec{B} = 4i + 6j$ are parallel, for the reson that

$$\vec{B} = 2(2i + 3j) = 2\vec{A}$$



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Vectors in aspace :-

$\vec{A} = \vec{OA} = a\vec{i} + b\vec{j} + c\vec{k}$

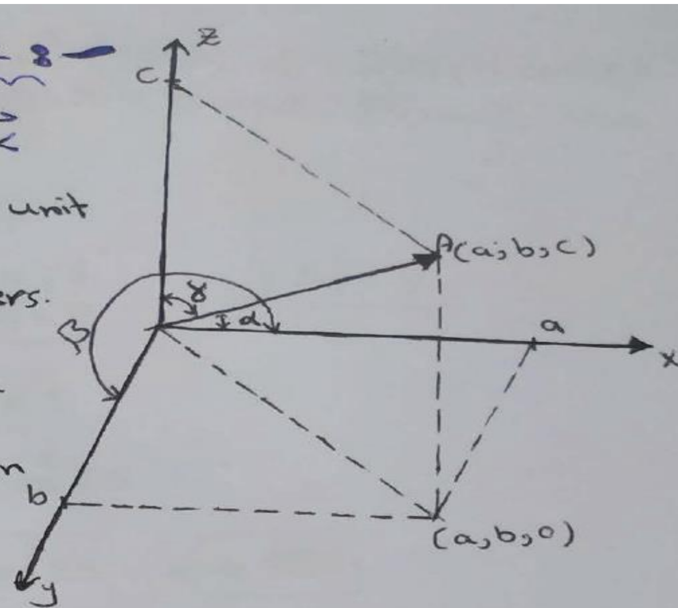
where :

$i, j, k$ :- are the fundamental unit vectors.

$a, b, c$ :- Direction numbers.

$\alpha, \beta, \gamma$ :- Direction angle

$\cos\alpha, \cos\beta, \cos\gamma$ :- Direction cosines.




* $\vec{U_A} = \dfrac{\vec{A}}{|A|}$

* Length of $\vec{A} = |A| = \sqrt{a^2 + b^2 + c^2}$

$$\vec{U_A} = \frac{ai + bj + ck}{|A|}$$

$$\vec{U_A} = \frac{a}{|A|} i + \frac{b}{|A|} j + \frac{c}{|A|} k$$

$$\cos\alpha = \frac{a}{|A|} \quad ; \quad \cos\beta = \frac{b}{|A|} \quad ; \quad \cos\gamma = \frac{c}{|A|}$$

$$\therefore \vec{U_A} = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$$

where

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

and $\vec{A} = |A| \cdot \vec{U_A}$

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EX(1):- Find a vector in $R^3$ (space) of length (5 units) that makes angles (70°) with the x-axis, (85°) with the y-axis

Answer:-

$\alpha = 70^\circ$ $\qquad B = 85^\circ$ $\qquad |A| = 5$

$\gamma = ?$ $\qquad$ Vector $\vec{A} = ?$

$$\cos^2\alpha + \cos^2 B + \cos^2\gamma = 1$$

$$\cos^2\gamma = 1 - \cos^2 70 - \cos^2 85$$

$$\cos\gamma = \sqrt{1 - \cos^2 70 - \cos^2 85} = \boxed{0.935}$$

$$\vec{A} = |A| \cdot (\cos\alpha\, i + \cos B\, j + \cos\gamma\, k)$$

$$= 5 * (\cos 70^\circ\, i + \cos 85^\circ\, j + 0.935\, k)$$

$$= \boxed{5 * (1.7\, i + 0.435\, j + 0.935\, k)}$$

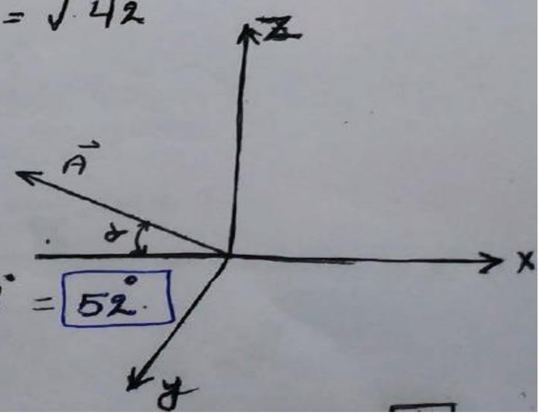
EX(2):- Find the acute angle between the x-axis and the vector $\vec{A} = -4i + 5j + k$ ?

Answer:-

$$|A| = \sqrt{(-4)^2 + (5)^2 + (1)^2} = \sqrt{42}$$

$$\cos\alpha = \frac{a}{|A|} = \frac{-4}{\sqrt{42}} = 0.617$$

$$\alpha = \cos^{-1}(0.617) \cong 128^\circ$$




∴ The required angle is $180^\circ - 128^\circ = \boxed{52^\circ}$

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## Definition

**(1) Algebrac addition :-**

If $\vec{A} = a_1 i + b_1 j$ and $\vec{B} = a_2 i + b_2 j$

Then $\vec{A} + \vec{B} = (a_1 + a_2) i + (b_1 + b_2) j$

**(2) Subtraction :-**

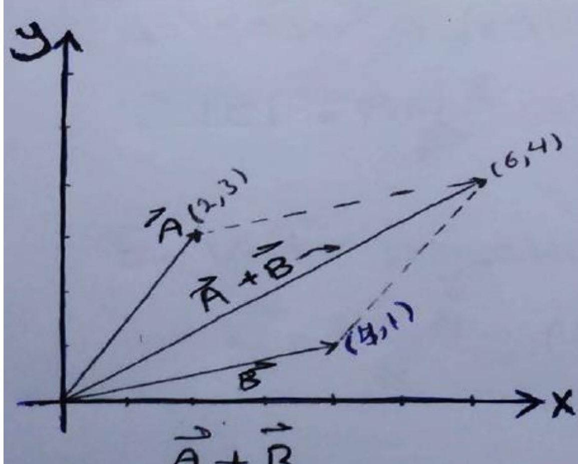
$$\vec{A} - \vec{B} = (a_1 - a_2) i + (b_1 - b_2) j$$

**EX(1) :-** If $\vec{A} = 2i + 3j$ and $\vec{B} = 4i + j$, then find $\vec{A} + \vec{B}$ and $\vec{B} - \vec{A}$ and sketch ?

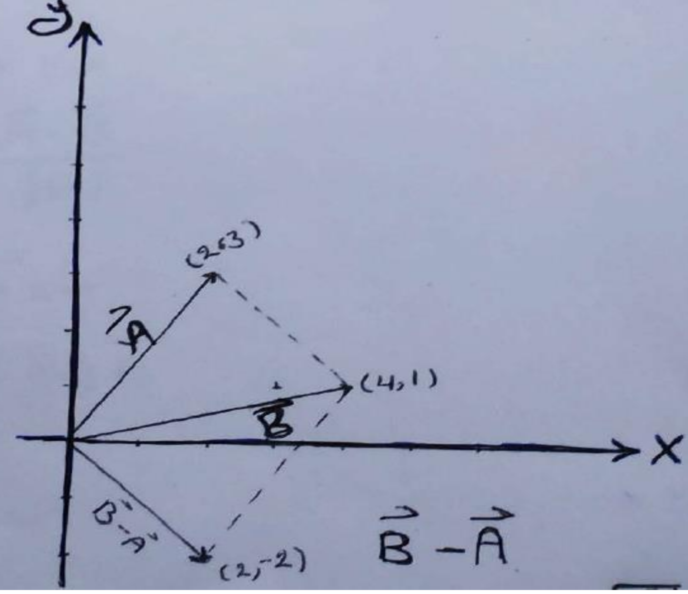
**Answer :-**

$$\vec{A} + \vec{B} = (2i + 3j) + (4i + j)$$
$$= (2 + 4) i + (3 + 1) j$$
$$= \boxed{6i + 4j}$$

$$\vec{B} - \vec{A} = (4i + j) - (2i + 3j)$$
$$= (4 - 2) i + (1 - 3) j$$
$$= \boxed{2i - 2j}$$




$\vec{A} + \vec{B}$




$\vec{B} - \vec{A}$

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* Scalar product (Dot-product) :-

Let $\vec{A} = a_1 i + a_2 j + a_3 K$ and $\vec{B} = b_1 i + b_2 j + b_3 K$

* $\boxed{\vec{A} \cdot \vec{B} = |A| \cdot |B| \cdot \cos\theta}$

where $\theta$ is the angle between $\vec{A}$ and $\vec{B}$

Properties :-

1- $\vec{A} \cdot \vec{A} = |A|^2$

2- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = a_1 b_1 + a_2 b_2 + a_3 b_3$

3- $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|A| \cdot |B|} = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}$

4- $\vec{A} \perp \vec{B} \iff \vec{A} \cdot \vec{B} = 0$ [Orthogonal vectors]

5- $ai + bj \perp bi - aj$

6- projection :-

a- Scalar projection :-

$$|C| = \text{Proj}_{\vec{B}} \vec{A} = \frac{\vec{A} \cdot \vec{B}}{|B|}$$

b- Vector projection :-

$$\vec{C} = \vec{\text{Proj}}_{\vec{B}} \vec{A} = \left(\frac{A \cdot B}{B \cdot B}\right) \vec{B}$$

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**EX(1):-** Find the angle $\theta$ between $\vec{A} = i - 2j - 2K$ and $\vec{B} = 6i + 3j + 2K$ ?

**Answer:-**

$$A \cdot B = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = \boxed{-4}$$

$$|A| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = \boxed{3}$$

$$|B| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = \boxed{7}$$

$$|A| \cdot |B| = 3 * 7 = \boxed{21}$$

$$\therefore \cos\theta = \frac{A \cdot B}{|A| \cdot |B|} = \frac{-4}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{-4}{21}\right)$$

$$\therefore \boxed{\theta \approx 101^\circ}$$

**EX(2):-** Show that the two vectors $\vec{A} = 3i - 2j + K$ and $\vec{B} = 2j + 4K$ are orthogonal ?

**Answer:-**

$$A \cdot B = (3)(0) + (-2)(2) + (1)(4)$$
$$= 0 - 4 + 4 = 0$$

$$\because A \cdot B = 0$$

$\therefore$ The two vectors $\vec{A}$ and $\vec{B}$ are orthogonal

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EX(3):- Find the vector projection of $(\vec{B} = 6i + 3j + 2k)$ onto $(\vec{A} = i - 2j - 2k)$ and the scalar component of $\vec{B}$ in the direction of $\vec{A}$?

Answer:- The vector projection is

$$\text{proj}_{\vec{A}}\vec{B} = \left(\frac{A \cdot B}{A \cdot A}\right)\vec{A}$$

$$= \left(\frac{(1)(6) + (-2)(3) + (-2)(2)}{(1)^2 + (-2)^2 + (-2)^2}\right)(i - 2j - 2k)$$

$$= \frac{6 - 6 - 4}{1 + 4 + 4}(i - 2j - 2k)$$

$$= \frac{-4}{9}(i - 2j - 2k)$$

$$= \boxed{\frac{-4}{9}i + \frac{8}{9}j + \frac{8}{9}k}$$

and the scalar component is

$$\text{proj}_{\vec{A}}\vec{B} = \frac{\vec{A} \cdot \vec{B}}{|A|}$$

$$= \frac{(1)(6) + (-2)(3) + (-2)(2)}{\sqrt{(1)^2 + (-2)^2 + (-2)^2}}$$

$$= \frac{-4}{\sqrt{9}} = \boxed{\frac{-4}{3}}$$

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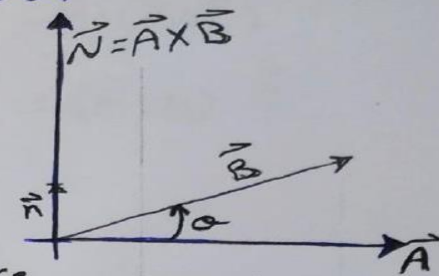
## Vector Product (Cross Product):-

$$* \boxed{\vec{N} = \vec{A}\times\vec{B} = \vec{n}\,|A|\,|B|\,\sin\theta}$$




where $\vec{n}$ is the normal unit vector normal to $\vec{A}$ and $\vec{B}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where, $\vec{A} = a_1 i + a_2 j + a_3 k$
$\vec{B} = b_1 i + b_2 j + b_3 k$

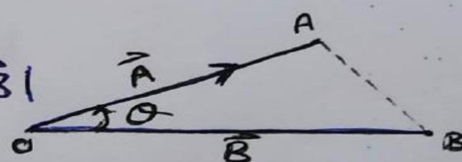
**Properties:-**

1- $\vec{A}\times\vec{A} = 0$

2- $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

3- $\vec{A} \parallel \vec{B} \iff \vec{A}\times\vec{B} = 0$

4- Area of $\triangle OAB = \frac{1}{2}|\vec{A}\times\vec{B}|$




**EX(1):-** Find $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$ if $(\vec{A} = 2i + j + k)$ and $(\vec{B} = -4i + 3j + k)$ ?

**Answer:-**

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= [(1*1)-(3*1)]\,i - [(2*1)-(-4*1)]\,j + [(2*3)-(-4*1)]\,k$$

$\boxed{\vec{A}\times\vec{B} = -2i - 6j + 10k}$ but $\boxed{\vec{B}\times\vec{A} = -\vec{A}\times\vec{B} = 2i + 6j - 10k}$ $\boxed{9}$

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## Triple product:-

(1) Scalar triple product $(\vec{A},\vec{B},\vec{C}) = \vec{A}\cdot(\vec{B}\times\vec{C})$

$$= (\vec{A}\times\vec{B})\cdot\vec{C}$$

Note:- Volume of box is

$$\boxed{V = |\vec{A}\cdot\vec{B}\times\vec{C}|}$$

Note:- Volume of pyramid (Tetrahedron)

$$\boxed{V = \frac{1}{6}|\vec{A}\cdot\vec{B}\times\vec{C}|}$$

(2) Vector triple product

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

Note:- $i\cdot i = j\cdot j = k\cdot k = 1$

$i\times i = j\times j = k\times k = 0$

$i\cdot j = j\cdot k = k\cdot i = 0$

$i\times j = k$

$j\times k = i$

$k\times i = j$

اسم المادة : رياضيات
اسم التدريسي : : م.م زين العابدين علي
اسم التدريسي : : د حسين كاظم حلواص
المرحلة : الاولى
السنة الدراسية : 2023-2024
عنوان المحاضرة: المتجهات vectors

## H.W No.1

1- Find the length and direction of each vector and the angle it makes with the positive X-axis
(a) $i + j$ (b) $\sqrt{3}\, i + j$ (c) $5i + 12j$

2- Find a unit vector in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

3- Find a vector 6 units long in the direction of $\vec{A} = 2i + 2j - K$.

4- Find the length and direction of $A \times B$ and $B \times A$
(a) $\vec{A} = 2i - 2j - K$ $\vec{B} = i + j + K$
(b) $\vec{A} = 2i$ $\vec{B} = -3j$

5- Find the area of the tringle whose vertices are $A(1, -1, 0)$ ; $B(2, 1, -1)$ and $C(-1, 1, 2)$

6- If $\vec{A} = 2i - j$ and $\vec{B} = i + 3j - 2K$, find $A \times B$ then calculate $(A \times B) \cdot A$

7- Let $\vec{A} = 5i - j + k$ ; $B = j - 5K$ and $\vec{C} = -15i + 3j - 3K$
which pairs of vector are (a) Perpendicular
(b) Parallel ?

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## ----- نهاية محاضرة " المتجهات Vectors " -----