AL-MUSTAQBAL UNIVERSITY
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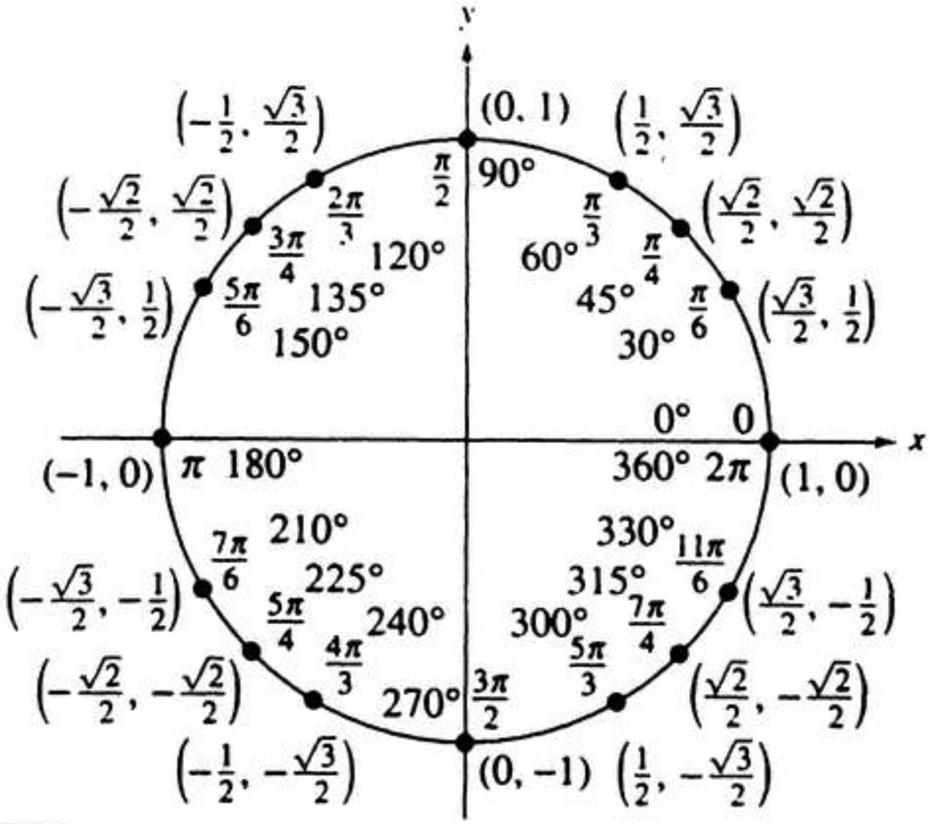
Department of BioMedical TEngineering



Subject Physics.

Class 1

Lecture 1 (Units, Trigonometry and vectors.)

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A **trigonometry formula** is a formula that is used to represent relationships between the **parts of a triangle** including the **side lengths**, **angles** and the **area**.

The definitions of the trigonometric functions and the inverse trigonometric functions, as well as the Pythagorean theorem, can be applied to any right triangle, regardless of whether its sides correspond to x - and y - coordinates. These results from trigonometry are useful in converting from rectangular coor- dinates to polar coordinates, or vice versa,

Basic Trigonometric Identities

Quotient identities:

$$\tan(A) = \frac{\sin(A)}{\cos(A)} \qquad \cot(A) = \frac{\cos(A)}{\sin(A)}$$

Even/Odd id entities:

$$\cos(-A) = \cos(A)$$
$$\sec(-A) = \sec(A)$$

Even functions

$$\sin(-A) = -\sin(A)$$
$$\csc(-A) = -\csc(A)$$

Odd functions

$$\tan(-A) = -\tan(A)$$
$$\cot(-A) = -\cot(A)$$

Odd functions

Reciprocal Identities:

$$\csc(A) = \frac{1}{\sin(A)} \qquad \sec(A) = \frac{1}{\cos(A)} \qquad \cot(A) = \frac{1}{\tan(A)}$$

$$\sin(A) = \frac{1}{\csc(A)} \qquad \cos(A) = \frac{1}{\sec(A)} \qquad \tan(A) = \frac{1}{\cot(A)}$$

Pythagorean Identities:

$$\sin^2(A) + \cos^2(A) = 1$$

$$\tan^2(A) + 1 = \sec^2(A) \qquad 1 + \cot^2(A) = \csc^2(A)$$

Radius of the circle is 1.

x = cos(θ) $-1 \le \cos(\theta) \le 1$

y = sin(θ) $-1 \le \sin(\theta) \le 1$

Pythagorean Theorem: $x^2 + y^2 = 1$

This gives the identity: $\cos^2(\theta) + \sin^2(\theta) = 1$

Zeros of sin(θ) are $n\pi$ where *n* is an integer.

Zeros of cos(θ) are $\frac{\pi}{2} + n\pi$ where *n* is an integer.

Trigonometric Identities Summation & Difference Formulas

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

Trigonometric Identities Double Angle Formulas

$$\sin(2A) = 2\sin(A)\cos(A)$$

$$\cos(2A) = \cos^2(A) - \sin^2(A) = 1 - 2\sin^2(A) = 2\cos^2(A) - 1$$

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

Trigonometric Identities
Half Angle Formulas

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$$

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}}$$

The quadrant of $\frac{A}{2}$ determines the sign.

Introduction:

Scalars and Vectors:

A quantity which is completely specified by a certain number associated with a suitable unit without any mention of direction in space is known as scalar. Examples of scalar are **time, mass, length, volume, density, temperature, energy, distance, speed etc**. The number describing the quantity of a particular scalar is known as its magnitude. The scalars are added subtracted, multiplied and divided by the usual arithmetical laws. A quantity which is completely described only when both their magnitude and direction are specified is known as vector. Examples of vector are force, velocity, **acceleration, displacement, torque, momentum, gravitational force, electric and magnetic intensities etc**. A vector is represented by a Roman letter in bold face and its magnitude, by the same letter in italics. Thus V means vector and V is magnitude.

Types of Vectors:

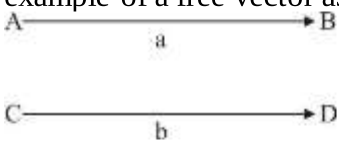
1. **Unit Vector:**

A vector whose magnitude is unity i.e., 1 and direction along the given vector is called a unit Vector. If $\vec{a}$ is a vector then a unit vector in the direction of $\vec{a}$, denoted by $\hat{a}$ (read as a cap), is given as,

$$\hat{a} = \frac{\vec{a}}{|a|} \quad \text{or} \quad \vec{a} = |a|\,\hat{a}$$

2. **Free Vector:**

A vector whose position is not fixed in space. Thus, the line of action of a free vector can be shifted parallel to itself. Displacement is an example of a free vector as shown in figure 1:



3. **Localized or Bounded Vectors:**

A vector which cannot be shifted parallel to itself, i.e., whose line of action is fixed is called a localized or bounded vector. Force and momentum are examples of localized vectors.

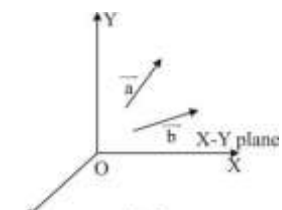


Fig. 2

4. **Coplanar Vectors:**

The vectors which lies in the same plane are called coplanar vectors, as shown in Fig. 2.

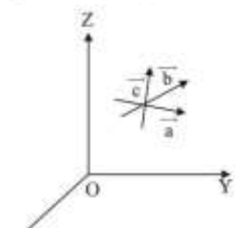


Fig. 3

5. **Concurrent Vectors:**

The vectors which pass through the common point are called concurrent vectors. In the figure no.3 vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are called concurrent as they pass through the same point.

6. **Negative of a Vector:**

The vector which has the same magnitude as the vector $\vec{a}$ but opposite in direction to $\vec{a}$ is called the negative to $\vec{a}$. It is represented by $-\vec{a}$. Thus of $\overrightarrow{AB} = -\vec{a}$ then $\overrightarrow{BA} = -\vec{a}$

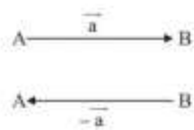


Fig. 4

7. Null or Zero Vector:

It is a vector whose magnitude is zero. We denote the null vector by O. The direction of a zero vector is arbitrary.

The vectors other than zero vectors are proper vectors or non-zero vectors.

8. Equal Vectors:

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude and direction. If $\vec{a}$ and $\vec{b}$ are equal vectors then $\vec{a} = \vec{b}$

9. Parallel and Collinear Vectors:

The vectors $\vec{a}$ and $\vec{b}$ are parallel if for any real number n,

$\vec{a} = n\,\vec{b}$. If

(i) $n > 0$ then the vectors $\vec{a}$ and $\vec{b}$ have the same direction.

(ii) $n < 0$ then $\vec{a}$ and $\vec{b}$ have opposite directions.

Now, we can also define collinear vectors which lie along the same straight line or having their directions parallel to one another.

10. Like and Unlike Vectors:

The vectors having same direction are called like vectors and those having opposite directions are called unlike vectors.

11. Position Vectors (PV):

If vector $\vec{OA}$ is used to specify the position of a point A relative to another point O. This $\vec{OA}$ is called the position vector of A referred to O as origin. In the figure 4 $\vec{a} = \vec{OA}$ and $\vec{OB} = \vec{b}$ are the position vector (P.V) of A and B respectively. The vector $\vec{AB}$ is determined as follows:

By the head and tail rules,

$\overline{OA} + \overline{AB} = \overline{OB}$

Or $\overline{AB} = \overline{OB} - \overline{OA} = \overline{b} - \overline{a}$

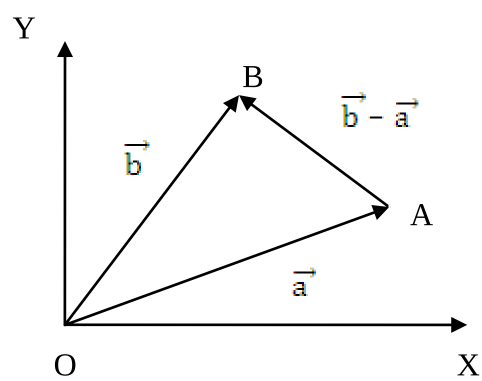


Fig. 5

Addition and Subtraction of Vectors:

1. Addition of Vectors:

Suppose $\vec{a}$ and $\vec{b}$ are any two vectors. Choose point A so that $\vec{a} = \vec{OA}$ and choose point C so that $\vec{b} = \vec{AC}$. The sum, $\vec{a} + \vec{b}$ of $\vec{a}$ and $\vec{b}$ is the vector is the vector $\vec{OC}$. Thus the sum of two vectors $\vec{a}$ and $\vec{b}$ is performed by the Triangle Law of addition.

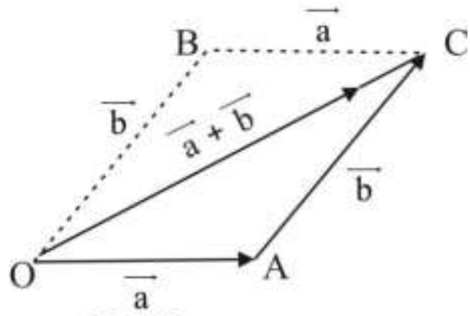


Fig. 6

2. Subtraction of Vectors:

If a vector $\vec{b}$ is to be subtracted from a vector $\vec{a}$, the difference vector $\vec{a} - \vec{b}$ can be obtained by adding vectors $\vec{a}$ and $-\vec{b}$.

The vector $-\vec{b}$ is a vector which is equal and parallel to that of vector but its arrow-head points in opposite direction. Now the vectors $\vec{a}$ and $-\vec{b}$ can be added by the head-to-tail rule. Thus the line AC represents, in magnitude and direction, the vector $\vec{a} - \vec{b}$.

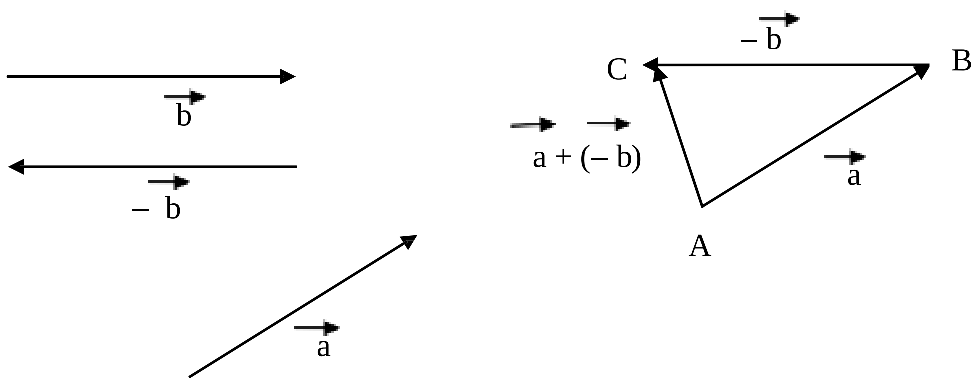


Fig . 7

Properties of Vector Addition:

i. Vector addition is commutative

i.e., $\vec{a} + \vec{b} = \vec{a} + \vec{b}$ where $\vec{a}$ and $\vec{b}$ are any two vectors.

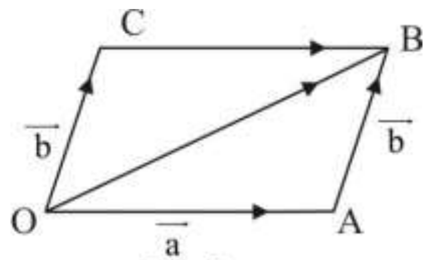


Fig. 8

(ii) Vectors Addition is Associative:

i.e. $(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c})$

where $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ are any three vectors.

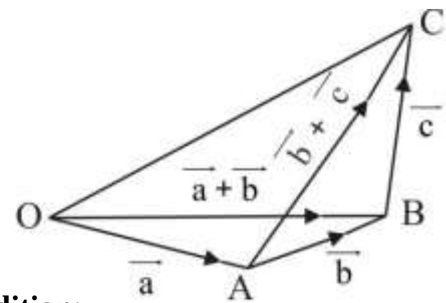


Fig.9

(iii) $\dot{O}$ is the identity in vectors addition:

For every vector $\overrightarrow{a}$

$\overrightarrow{a} + \overrightarrow{O} = \overrightarrow{a}$

Where $\dot{O}$ is the zero vector.

Remarks: Non-parallel vectors are not added or subtracted by the ordinary algebraic Laws because their resultant depends upon their directions as well.

Multiplication of a Vector by a Scalar:

If $\overrightarrow{a}$ is any vectors and K is a scalar, then $K\overrightarrow{a} = \overrightarrow{a}K$ is a vector with magnitude $|K|.|a|$ i.e., $|K|$ times the magnitude of $\overrightarrow{a}$ and whose direction is that of vector $\overrightarrow{a}$ or opposite to vector $\overrightarrow{a}$ according as K is positive or negative resp. In particular $\overrightarrow{a}$ and $-\overrightarrow{a}$ are opposite vectors.

Properties of Multiplication of Vectors by Scalars:

1. The scalar multiplication of a vectors satisfies

$$m(n\overrightarrow{a}) = (mn)\overrightarrow{a} = n(m\overrightarrow{a})$$

2. The scalar multiplication of a vector satisfies the distributive laws

i.e., $(m + n)\overrightarrow{a} = m\overrightarrow{a} + n\overrightarrow{a}$

and $\qquad m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Where m and n are scalars and $\vec{a}$ and $\vec{b}$ are vectors.

The Unit Vectors i, j, k (orthogonal system of unit Vectors):

Let us consider three mutually perpendicular straight lines OX, OY and OZ. These three mutually perpendicular lines determine uniquely the position of a point. Hence these lines may be taken as the co-ordinates axes with O as the origin.

We shall use i, j and k to denote the Unit Vectors along OX, OY and OX respectively.

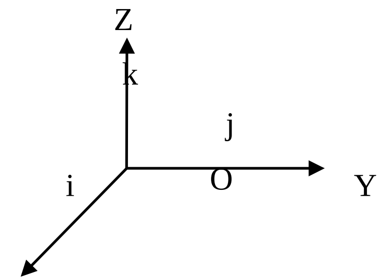


Fig. 9a

Representation of a Vector in the Form of Unit Vectors i, j and k.

Let us consider a vector $\vec{r} = \overrightarrow{OP}$ as shown in fig. 11. Then x i, y j and z k are vectors directed along the axes,

$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \overrightarrow{OA} + \overrightarrow{OB} \qquad \text{because}$$

and $\qquad \overrightarrow{OQ} = xi + yi$

Because $\qquad \overrightarrow{QP} = zk$

$\qquad \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$

and $\qquad \vec{r} = \overrightarrow{OP} = xi + yj + zk$

Here the real numbers x, y and z are the components of Vector $\vec{r}$ or the co-ordinates of point P in the direction of OX, OY and OZ respectively. The vectors xi, yj and zk are called the resolved parts of the vector $\vec{r}$ in the direction of the Unit vectors i, j and k respectively.

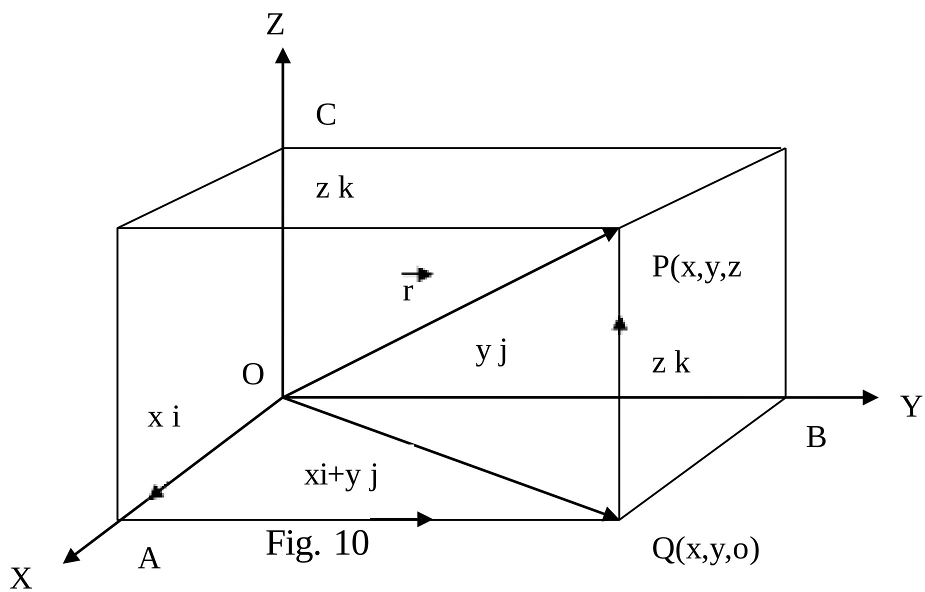


Fig. 10

Components of a Vector when the Tail is not at the Origin:

Consider a vector $\vec{r} = \vec{PQ}$ whose tail is at the point $P(x_1, y_1, z_1)$ and the head at the point Q (x_2, y_2, z_2). Draw perpendiculars PP′ and QQ′ on x-aixs.

$P'Q' = x_2 - x_1$ = x-component of $\vec{r}$

Now draw perpendiculars PP^o and QQ^o on y-axis.

Then $P^oQ^o = y_2 - y_1$ = y-component of $\vec{r}$

Similarly $z_2 - z_1$ = z-component of $\vec{r}$

Hence the vector $\vec{r}$ can be written as,

$\vec{r} = \vec{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Or, $\vec{r} = \vec{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

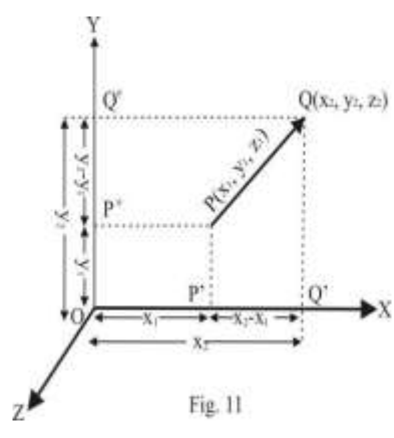


Fig. 11

Magnitude or Modulus of a Vector:

Suppose x, y and z are the magnitude of the vectors $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ as shown in fig. 10.

In the right triangle OAQ, by Pythagorean Theorem

$$OQ^2 = x^2 + y^2$$

Also in the right triangle OQP, we have

$$OP^2 = OQ^2 + QP^2$$

$$OP^2 = x^2 + y^2 + z^2$$

Or $$|\overline{r}| = |OP| = \sqrt{x^2 + y^2 + z^2}$$

Thus if $\overline{r} = \overline{PQ} = xi + yj + zk$

Then , its magnitude is

$$|\overline{r}| = \sqrt{x^2 + y^2 + z^2}$$

If $\overline{r} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Then $|\overline{r}| = \sqrt{(x_2 - x_1)2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 1

Find the magnitude of the vector

$$|\overline{u}| = \frac{3}{5}i - \frac{2}{5}j + \frac{2\sqrt{3}}{5}k$$

Solution:

$$|\overline{u}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 + \left(\frac{2\sqrt{3}}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + \frac{4}{25} + \frac{12}{25}} = \sqrt{\frac{25}{25}}$$

$$|\overline{u}| = 1$$

Note: Two vectors are equal if and only if the corresponding components of these vectors are equal relative to the same co-ordinate system.

Example 2:

Find real numbers x, y and z such that

$xi + 2yj – zk + 3i – j = 4i + 3k$

Solution:

Since $(x + 3)i + (2y – 1)j + (–z)k = 4i + 3k$

Comparing both sides,we get

$x + 3 = 4,\ 2y – 1 = 0,\quad –z = 3$

$x = 1,\ \ y = \frac{1}{2}\ ,\ \ z = –3$

Note 2: If $\vec{r_1} = x_1i + y_1j + z_1k$

$\vec{r_2} = x_2i + y_2j + z_2k$

Then the sum vector =

$$\vec{r_1} + \vec{r_2} = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$$

Or $$\vec{r_1} + \vec{r_2} = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$$

Example 3

$\vec{a} = 3i - 2j + 5k$ and $\vec{b} = -2i - j + k$.

Find $2\vec{a} - 3\vec{b}$ and also its unit vector.

Solution:

$2\vec{a} - 3\vec{b} = 2(3i – 2j + 5k) – 3(–2i – j + k)$

$= 6i – 4j + 10k + 6i + 3j – 3k$

$= 12i – j + 7k$

If we denote $2\vec{a} - 3\vec{b} = \vec{c}$, then $\vec{c} = 12i - j + 7k$

and $|\vec{c}| = \sqrt{12^2 + (-1)^2 + 7^2} = \sqrt{144 + 1 + 49} = \sqrt{194}$

Therefore, $$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{12i - j + 7k}{\sqrt{194}}$$

$$\hat{c} = \frac{12}{\sqrt{194}}i - \frac{1}{\sqrt{194}}j + \frac{1}{\sqrt{194}}k$$

Note 3: Two vectors $\vec{r_1} = x_1i + y_1j + z_1k$ and $\vec{r_2} = x_2i + y_2j + z_2k$ are parallel if and only if $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.

Direction Cosines:

Let us consider that the vector $\dot{r} = \dot{OP}$ which makes angles α, β and γ with the coordinate axes OX, OY and OZ respectively. Then Cos α, Cos β and Cos γ are called the direction cosines of the vector $\dot{OP}$. They are usually denoted by l, m and n respectively.

If $\overline{OP} = \overline{r} = xi + yj + zk$, then x, y and z are defined as the direction ratios of the vector $\dot{r}$ and $\overline{r} = \sqrt{x^2 + y^2 + z^2}$. Since the angles A, B and C are right angles (by the fig. 11), so in the right triangles.

OAP, OBP and OCP the direction cosines of $\dot{r}$ can be written as,

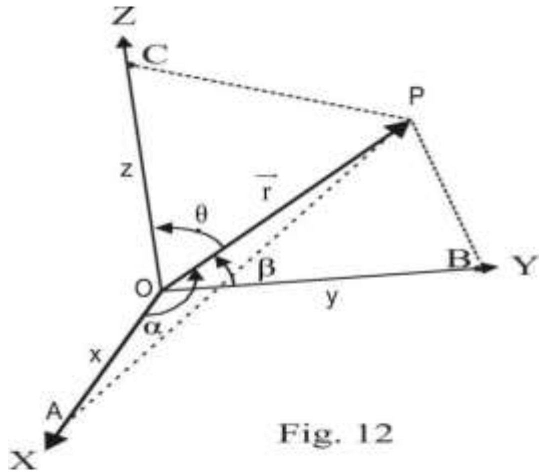


Fig. 12

$$l = \cos\alpha = \frac{x}{|r|} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$m = \cos\beta = \frac{y}{|r|} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

and $$n = \cos\gamma = \frac{z}{|r|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Note 1: Since the unit vector $\hat{r} = \frac{\dot{r}}{|r|} = \frac{xi + yj + zk}{|r|}$

$$\hat{r} = \frac{x}{|r|}i + \frac{y}{|r|}j + \frac{z}{|r|}k$$

$$\hat{r} = \text{Cos}\,\alpha\ i + \text{Cos}\,\beta\ j + \text{Cos}\,\gamma\ k$$

Or $$\hat{r} = li + mj + nk$$

Therefore the co-efficient of i, j and k in the unit vector are the direction cosines of a vector.

Note 2: $$l^2 + m^2 + n^2 = \frac{x^2}{|r|^2} + \frac{y^2}{|r|^2} + \frac{z^2}{|r|^2}$$

$$= \frac{x^2 + y^2 + z^2}{|r|^2} = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1$$

Example 5:

Find the magnitude and direction cosines of the vectors 3i + 7j – 4k, i – 5j – 8k and 6i – 2j + 12k.

Solution:

Let $\vec{a} = 3i + 7j - 4k$

$\vec{b} = i - 5j - 8k$

$\vec{c} = 6i - 2j + 12k$

Now $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3i + 7j - 4k}{\sqrt{74}}$

$= \frac{3}{\sqrt{74}} i + \frac{7}{\sqrt{74}} j - \frac{4}{\sqrt{74}} k$

So the direction cosines of $\vec{a}$ are: $\frac{3}{\sqrt{74}}, \frac{7}{\sqrt{74}}, -\frac{4}{\sqrt{74}}$

Similarly the direction cosines of $\vec{b}$ are: $\frac{1}{\sqrt{90}}, -\frac{5}{\sqrt{90}}, -\frac{8}{\sqrt{90}}$

and the direction cosines of $\vec{c}$ are: $\frac{6}{\sqrt{184}}, -\frac{2}{\sqrt{184}}, \frac{12}{\sqrt{184}}$

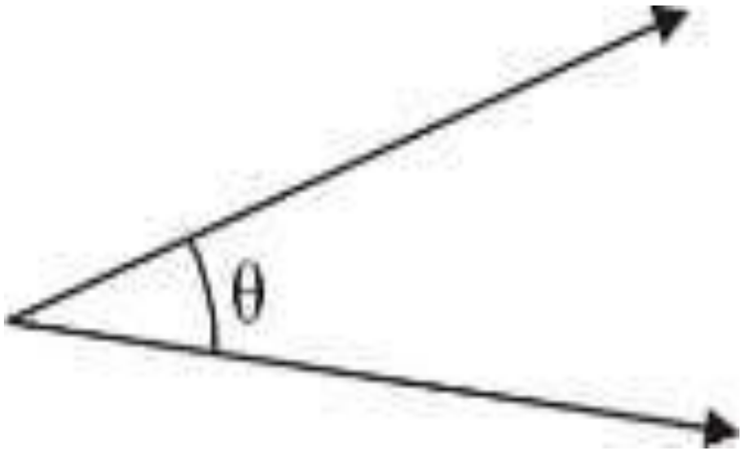
Exercise 1.1

1. If $\vec{a} = 3i - j - 4k$, $b = -2i + 4j - 3k$ and $\vec{c} = i + 2j - k$. Find unit vector parallel to $3\vec{a} - 2\vec{b} + 4\vec{c}$.
2. Find the vector whose magnitude is 5 and which is in the direction of the vector 4i – 3j + k.
3. For what value of m, the vector 4i + 2j – 3k and mi – j + $\sqrt{3}$k have same magnitude?
4. Find the lengths of the sides of a triangle, whose vertices are A = (2, 4, –1), B = (4, 5, 1), C = (3, 6, –3) and show that the triangle is right angled.

Product of Vectors:

1. Scalar Product of two Vectors:

If a and b are non-zero vectors, and θ is the angle between them, then the



scalar product of a and b is denoted by a . b and read as a dot b . It is defined by the relation

$$a \cdot b = |a|\,|b| \cos\theta$$

If either a or b is the zero vector, then a . b = 0

…………..

Remarks:

i. The scalar product of two vectors is also called the dot product because the "." used to indicate this kind of multiplication. Sometimes it is also called the inner product.

ii. The scalar product of two non-zero vectors is zero if and only if they are at right angles to each other. For $a \cdot b = 0$ implies that $\cos\theta = 0$, which is the condition of perpendicularity of two vectors.

Deductions:

From the definition (1) we deduct the following:

i. If a and b have the same direction, then

$\theta = 0^o \Rightarrow \cos 0^o = 1$

$$a \cdot b = |\vec{a}|\ |\vec{b}|$$

ii. If a and b have opposite directions, then

$\theta = \pi \quad \Rightarrow \cos\pi = -1$

$$\therefore \quad a \cdot b = -|\vec{a}|\ |\vec{b}|$$

Iii $a \cdot b$ will be positive if $0 \leq \theta < \frac{\pi}{2}$

and negative if, $\frac{\pi}{2} < \theta \leq \pi$

iv. The dot product of a and b is equal to the product of magnitude of a and the projection of b on a. This illustrate the geometrical meaning of $a \cdot b$. In the fig. 15 $|\vec{b}|\cos\theta$ is the projection of b on a.

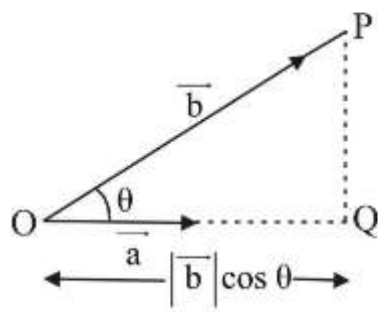


Fig. 15

v. From the equation (1)

$$b \cdot a = |\vec{b}|\ |\vec{a}| \cos\theta$$

$$= |\vec{a}|\ |\vec{b}| \cos\theta$$

$$b \cdot a = a \cdot b$$

Hence the dot product is commutative.

Corollary 1:

If $\vec{a}$ be a vector, then the scalar product $\vec{a}\cdot\vec{a}$ can be expressed with the help of equation (1) as follows:

$$\vec{a}\cdot\vec{a} = |\vec{a}|\,|\vec{a}|\,\text{Cos } 0^{o} = |\vec{a}|^{2}$$

Or $$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} \qquad (2)$$

This relation gives us the magnitude of a vector in terms of dot product.

Corollary 2:

If i, j and k are the unit vectors in the directions of X – , Y – and Z – axes, then from eq. (2)

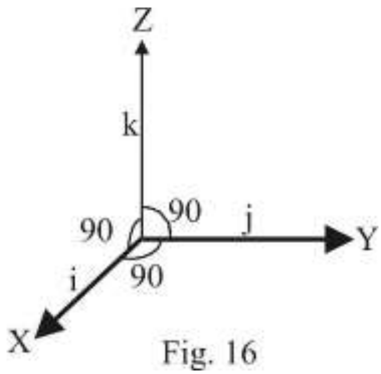


Fig. 16

$i^2 = i.i = |i|\ |i|\ \text{Cos}0^{o}$

$i^2 = 1$

so $i^2 = j^2 = k^2 = 1$

and $i\,.\,j = j\,.\,i = 0$ Because $\text{Cos}90^{o} = 0$

$i\,.\,k = k\,.\,i = 0$

$k\,.\,i = i\,.\,k = 0$

Corollary 3:

(Analytical expression of $\vec{a}\cdot\vec{b}$)

Scalar product of two vectors in terms of their rectangular components.

For the two vectors

$$\vec{a} = a_1i + a_2j + a_3k$$

and $$\vec{b} = b_1i + b_2j + b_3k$$

the dot product is given as,

$$\vec{a}\cdot\vec{b} = (a_1i + a_2j + a_3k)\,.\,(b_1i + b_2j + b_3k)$$

$$= a_1b_1 + a_2b_2 + a_3b_3 \text{ as } i^2 = j^2 = k^2 = 1$$

$$\text{and } i\,.\,j = j\,.\,k = k\,.\,i = 0$$

Also $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $a_1b_1 + a_2b_2 + a_3b_3 = 0$

Example 6:

If $\vec{a} = 3i + 4j – k$, $\vec{b} = –2i + 3j + k$ find $\vec{a}\cdot\vec{b}$

Solution:

$$\vec{a}\cdot\vec{b} = (3i + 4j – k)\,.\,(–2i + 3j + k)$$

$$= –6 + 12 –1$$

$$= 5$$

Where $|\bar{a}||\bar{b}|\sin\theta$ is the magnitude of $\bar{c}$ and $\hat{n}$ is the Unit Vector in the direction of $\bar{c}$. The direction of $\bar{c}$ is determined by the right hand rule.

The vector product is also called the „cross product‟ or „Outer product‟ of the vectors.

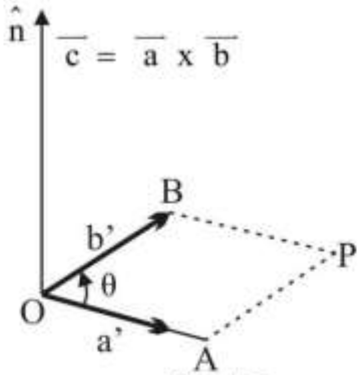


Fig. 17

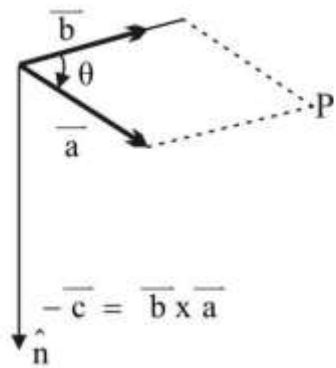


Fig. 18

Remarks:

If we consider $\bar{b} \times \bar{a}$, then $\bar{b} \times \bar{a}$ would be a vector which is opposite in the direction to $\bar{a} \times \bar{b}$.

Hence $\bar{a} \times \bar{b} = -\,\bar{b} \times \bar{a}$

Which gives that $\bar{a} \times \bar{b} \neq \bar{b} \times \bar{a}$ in general

Hence the vector product is not commutative.

Deductions:

The following results may be derived from the definition.

i. The vector product of two non-zero vectors is zero if $\bar{a}$ and $\bar{b}$ are parallel, the angle between $\bar{a}$ and $\bar{b}$ is zero. $\sin 0^o = 0$, Hence $\vec{a} \times \vec{b} = 0$.

For $\bar{a} \times \bar{b} = 0$ implies that $\sin\theta = 0$ which is the condition of parallelism of two vectors. In particular $\vec{a} \times \vec{a} = 0$. Hence for the unit vectors i, j and k,

$$i \times i = j \times j = k \times k = 0$$

ii. If $\bar{a}$ and $\bar{b}$ are perpendicular vectors, then $\bar{a} \times \bar{b}$ is a vector whose magnitude is $|\bar{a}||\bar{b}|$ and whose direction is such that the vectors a, b, a x b form a right-handed system of three mutually perpendicular

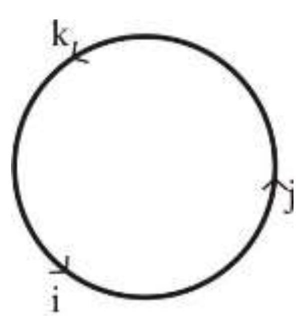


Fig. 19

vectors. In particular i x j = (1) (1) Sin 90° k (k being perpendicular to i and j) = k

Similarly j x i = −k, i x k = −j, k x j = −i

Hence the cross product of two consecutive unit vectors is the third unit vector with the plus or minus sign according as the order of the product is anti-clockwise or clockwise respectively.

iii. Since $\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \text{Sin } \theta$ …… (2)

Which is the area of the parallelogram whose two adjacent sides are $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$.

Hence, area of parallelogram OABC = $\left|\vec{a} \times \vec{b}\right|$

and area of triangle OAB = $\frac{1}{2}\left|\vec{a} \times \vec{b}\right|$

If th vertices of a parallelogram are given, then

area of parallelogram OABC = $\left|\overline{OA} \times \overline{OB}\right|$

and, area of triangle OAB = $\frac{1}{2}\left|\overline{OA} \times \overline{OB}\right|$

iv. If n is the unit vector in the directions of $\vec{c} = \vec{a} \times \vec{b}$ then

$$n = \frac{\vec{c}}{|c|} = \frac{\vec{a} \times \vec{b}}{|a \times b|}$$

or $$n = \frac{a \times b}{|a||b| \text{Sin } \theta}$$

from equation (2) we also find.

$$\text{Sin } \theta = \frac{\left|\vec{a} \times \vec{b}\right|}{|a|\,|b|}$$

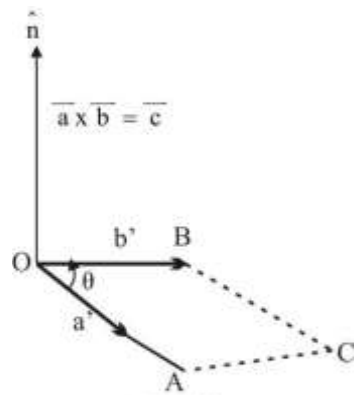


Fig. 20

Rectangular form of $a \times b$

(Analytical expression of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ **)**

If $a = a_1i + a_2j + a_3k$

and $b = b_1i + b_2j + b_3k$

then $a \times b = (a_1i + a_2j + a_3k) \times (b_1i + b_2j + b_3k)$

$= (a_1b_2k - a_1b_3j - a_2b_1k + a_2b_3j + a_3b_1j - a_3b_2j)$

$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$

This result can be expressed in determinant form as

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 11:

If $\vec{a} = 2i + 3j + 4k$ $\vec{b} = I - j + k$, Find

(i) $\vec{a} \times \vec{b}$

(ii) Sine of the angle between these vectors.

(iii) Unit vector perpendicular to each vector.

Solution:

(i) $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = i(3 + 4) - j(2 - 4) + k(-2 - 3)$$
$$= 7i + 2j - 5k$$

(ii) $$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{7^2 + 2^2 + (-5)^2}}{\sqrt{2^2 + 3^2 + 4^2}.\sqrt{1^2 + (-1)^2 + 1^2}}$$

$$= \frac{\sqrt{78}}{\sqrt{29}\sqrt{3}}$$

$$\sin\theta = \sqrt{\frac{26}{29}}$$

(iii) If $\hat{n}$ is the unit vector perpendicular to $\vec{a}$ and $\vec{b}$ then

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{7i + 2j - 5k}{\sqrt{78}}$$

Example 12:

$\vec{a} = 3i + 2k$, $\vec{b} = 4i + 4j - 2k$

$\vec{c} = i - 2j + 3k$, $\vec{d} = 2i - j + 5k$

Compute $(\vec{d} \times \vec{c}).(\vec{a} - \vec{b})$

Solution:

$$\vec{d} \times \vec{c} = \begin{vmatrix} i & j & k \\ 2 & -1 & 5 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= i(-3 + 10) - j(6 - 5) + k(-4 + 1)$$
$$= 7i - j - 3k$$

Also $\quad a - b = -i - 4j + 4k$

Hence $(d \times c).(a - b) = (7i - j - 3k).(-i - 4j + 4k)$

$= -7 + 4 - 12$

$= -15$

Example 13:

Find the area of the parallelogram with adjacent sides,

$a = i - j + k$, and $b = 2j - 3k$

Solution:

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 2 & -3 \end{vmatrix}$$

$= i(3 - 2) - j(-3 - 0) + k(2 + 0)$

$= i + 3j + 2k$

Area of parallelogram $= |a \times b| = \sqrt{1 + 9 + 4}$

$= \sqrt{14}$ square unit.

Example 14:

Find the area of the triangle whose vertices are

A(0, 0, 0), B(1, 1, 1) and C(0, 2, 3)

Solution:

Since $\quad AB = (1 - 0, 1 - 0, 1 - 0)$

$= (1, 1, 1)$

and $\quad AC = (0 - 0, 2 - 0, 3 - 0)$

$AC = (0, 2, 3)$

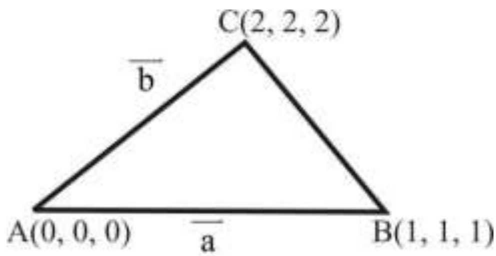


Fig. 21

$$AB \times AC = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix}$$

$= i(3 - 2) - j(3 - 0) + k(2 - 0)$

$= i - 3j + 2k$

Area of the triangle ABC $= \frac{1}{2}|AB \times AC| \quad = \frac{1}{2}\sqrt{1^2 + (-3)^2 + 2^2}$

$= \frac{\sqrt{14}}{2}$ square unit

Example 15:

Prove by the use of cross-product that the points

A(5, 2, −3), B(6, 1, 4), C(−2, −3, 6) and D(−3, −2, −1) are the vertices of a parallelogram.

Solution:

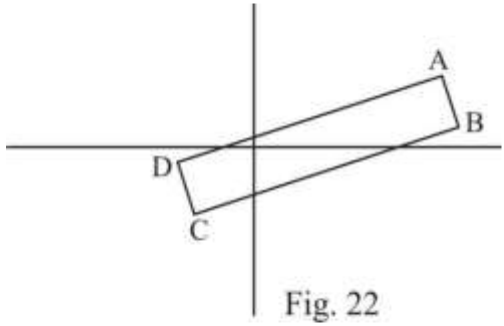


Fig. 22

Since $\vec{AB} = (1, -1, 7)$

$\vec{DC} = (+1, -1, +7)$

$\vec{BC} = (-8, -4, 2)$

and $\vec{AD} = (-8, -4, 2)$

$$\vec{AB} \times \vec{DC} = \begin{vmatrix} i & j & k \\ 1 & -1 & 7 \\ 1 & -1 & 7 \end{vmatrix}$$

$$= i(-7-7) - j(+7-7) + k(1-1)$$

$\vec{AB} \times \vec{DC} = 0$, so, $\vec{AB}$ and $\vec{DC}$ are parallel.

Also
$$\vec{BC} \times \vec{AD} = \begin{vmatrix} i & j & k \\ -8 & -4 & 2 \\ -8 & -4 & 2 \end{vmatrix}$$

$$= i(0) - j(0) + k(0)$$

$\vec{BC} \times \vec{AD} = 0$, so, $\vec{BC}$ and $\vec{AD}$ are parallel.

Hence the given points are the vertices of a parallelogram.

Exercise

Q.1 Find $\vec{a} \cdot \vec{b}$ and $\vec{a} \times \vec{b}$

(i) $\vec{a} = 2i + 3j + 4k$ $\quad$ $\vec{b} = i - j + k$

(ii) $\vec{a} = i + j + k$ $\quad$ $\vec{b} = -5i + 2j - 3k$

(iii) $\vec{a} = -i - j - k$ $\quad$ $\vec{b} = 2i + j$

2. Show that the vectors $3i - j + 7k$ and $-6i + 3j + 3k$ are at right angle to each other.

3. Find the cosine of the angle between the vectors:

(i) $\vec{a} = 2i - 8j + 3k$ $\quad$ $\vec{b} = 4j + 3k$

(ii) $\vec{a} = i + 2j - k$ $\quad$ $\vec{b} = -j - 2k$

(iii) $\vec{a} = 4i + 2j - k$ $\quad$ $\vec{b} = 2i + 4j - k$

Q.4 If $\vec{a} = 3i + j - k$, $\vec{b} = 2i - j + k$ and $\vec{c} = 5i + 3k$, find $(2\vec{a} + \vec{b}) \cdot \vec{c}$.

Q.5 What is the cosine of the angle between $\overline{P_1P_2}$ and $\overline{P_3P_4}$
If $P_1(2,1,3)$, $P_2(-4, 4, 5)$, $P_3(0, 7, 0)$ and $P_4(-3, 4, -2)$?

Q.6 If $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$, prove that:

$$\vec{a} \cdot \vec{b} = \frac{1}{2}\left[\left|\vec{a} + \vec{b}\right|^2 - \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2\right]$$

Q7> Find $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$ if $\vec{a} = i + 2j + 3k$ and $\vec{b} = 2i - j + k$.

Q8 Prove that for every pair of vectors $\vec{a}$ and $\vec{b}$

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2$$

Q9 Find x so that $\vec{a}$ and $\vec{b}$ are perpendicular,Q

(i) $\vec{a} = 2i + 4j - 7k$ and $\vec{b} = 2i + 6j + xk$

(ii) $\vec{a} = xi - 2j + 5k$ and $\vec{b} = 2i - j + 3k$

Q.10 If $\vec{a} = 2i - 3j + 4k$ and $\vec{b} = 2j + 4k$

Find the component or projection of $\vec{a}$ along $\vec{b}$.

Q.11 Under what condition does the relation $(\vec{a} \cdot \vec{b})^2 = \vec{a}^2 \, \vec{b}^2$ hold for two vectors $\vec{a}$ and $\vec{b}$.

Q.12 If the vectors 3i + j – k and λ I – 4 j + 4 k are parallel , find value of λ.

Q.13 If $\vec{a} = i - 2j + k$, $\vec{b} = i + 2j - 4k$, $\vec{c} = 2i - 3j + k$ Evaluate:

(i) $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$ (ii) $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$

Q.14 If $\vec{a} = i + 3j - 7k$ and $\vec{b} = 5i - 2j + 4k$. Find:

(i) $\vec{a} \cdot \vec{b}$ (ii) $\vec{a} \times \vec{b}$

(iii) Direction cosines of $\vec{a} \times \vec{b}$

Q.15 Prove that for the vectors $\vec{a}$ and $\vec{b}$

(i) $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a} \cdot \vec{b}\right|^2 = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2$

(ii) $(\vec{a} - \vec{b}) \times (\vec{a} \times \vec{b}) = 2(\vec{a} \times \vec{b})$

Q.16 Prove that for vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$[\vec{a} \times (\vec{b} + \vec{c})] + [\vec{b} \times (\vec{c} + \vec{a})] + [\vec{c} \times (\vec{a} + \vec{b})] = 0$$

Q.17 Find a vector perpendicular to both the lines AB and CD, where A is (0, 2, 4), B is (3, –1, 2), C is (2, 0, 1) and D is (4, 2, 0.)