المحاضرة الثانية

Dimensional Analysis

2.1 Introduction

Any phenomenon is physical sciences and engineering can be described by the *fundamentals dimensions* mass, length, time, and temperature. Till the rapid development of science and technology the engineers and scientists depend upon the experimental data. But the rapid development of science and technology has created new mathematical methods of solving complicated problems, which could not have been solved completely by analytical methods and would have consumed enormous time. This mathematical method of obtaining the equations governing certain natural phenomenon by balancing the fundamental dimensions is called *(Dimensional Analysis)*. Of course, the equation obtained by this method is known as *(Empirical Equation)*.

2.2 Fundamentals Dimensions

The various physical quantities used by engineer and scientists can be expressed in terms of fundamentals dimensions are: Mass (M), Length (L), Time (T), and Temperature (θ). All other quantities such as area, volume, acceleration, force, energy, etc., are termed as " derived quantities".

2.3 Dimensional Homogeneity

An equation is called *"dimensionally homogeneous"* if the fundamentals dimensions have identical powers of [L T M] (i.e. length, time, and mass) on both sides. Such an equation be independent of the system of measurement (i.e. metric, English, or S.I.). Let consider the common equation of volumetric flow rate,

$$Q = A\ u$$

$$L^3T^{-1} = L^2\,LT^{-1} = L^3T^{-1}.$$

We see, from the above equation that both right and left hand sides of the equation have the same dimensions, and the equation is therefore dimensionally homogeneous.

Example -2.1-

a) Determine the dimensions of the following quantities in M-L-T system 1- force 2- pressure 3- work 4- power 5- surface tension 6- discharge 7- torque 8- momentum.

b) Check the dimensional homogeneity of the following equations

1- $u = \sqrt{\dfrac{2\,g(\rho_m - \rho)\Delta z}{\rho}}$ 2- $Q = \dfrac{8}{15} cd \tan\dfrac{\theta}{2}\sqrt{2\,g}Z_{\circ}^{\frac{5}{2}}$

Solution:

a)

1- $F = m.g\ (kg.m/s^2)$ $\equiv [MLT^{-2}]$

2- $P=F/A \equiv [(MLT^{-2})\ (L^{-2})]$ (Pa) $\equiv [ML^{-1}T^{-2}]$

3- $Work = F.L \equiv [(MLT^{-2})\ (L)]$ (N.m) $\equiv [ML^2T^{-2}]$

4- $Power = Work/time \equiv [(ML^2T^{-2})\ (T^{-1})]$ (W) $\equiv [ML^{-1}T^{-2}]$

5- $Surface\ tension = F/L \equiv [(MLT^{-2})\ (L^{-1})]$ (N/m) $\equiv [ML^{-1}T^{-2}]$

6- Discharge (Q) m^3/s $\equiv [L^3T^{-1}]$

7- Torque (Γ) = F.L $\equiv [(MLT^{-2})(L)]$ N.m $\equiv [ML^2T^{-2}]$

8- Moment = m.u L)] N.m $\equiv [ML^2T^{-2}]$

b) 1- $u = \sqrt{\dfrac{2\,g(\rho_m - \rho)\Delta z}{\rho}}$

L.H.S. u $\equiv [LT^{-1}]$

R.H.S. u $\equiv \left[\dfrac{LT^{-2}(ML^3)}{ML^{-3}}\right]^{1/2} \equiv [LT^{-1}]$

Since the dimensions on both sides of the equation are same, therefore the equation is dimensionally homogenous.

2- $Q = \dfrac{8}{15} cd \tan\dfrac{\theta}{2}\sqrt{2\,g}Z_{\circ}^{\frac{5}{2}}$

L.H.S. u $\equiv [L^3T^{-1}]$

R.H.S. $(LT^{-2})\ (L)^{5/2} \equiv [L^3T^{-1}]$

This equation is dimensionally homogenous.

2.4.1 Rayleigh's method (or Power series)

In this method, the functional relationship of some variable is expressed in the form of an exponential equation, which must be dimensionally homogeneous. If (y) is some function of independent variables (x_1, x_2, x_3, ...etc.), then functional relationship may be written as;

$$y = f(x_1, x_2, x_3, \ldots\ldots\ldots \text{etc.})$$

The dependent variable (y) is one about which information is required; whereas the independent variables are those, which govern the variation of dependent variables.

The Rayleigh's method is based on the following steps:-

1- First of all, write the functional relationship with the given data.

2- Now write the equation in terms of a constant with exponents i.e. powers a, b, c,...

3- With the help of the principle of dimensional homogeneity, find out the values of a, b, c, ... by obtaining simultaneous equation and simplify it.

4- Now substitute the values of these exponents in the main equation, and simplify it.

Example -2.2-

If the capillary rise (h) depends upon the specific weight (sp.wt) surface tension (σ) of the liquid and tube radius (r) show that:

$$h = r\phi\left(\frac{\sigma}{(sp.wt.)\,r^2}\right), \text{ where } \phi \text{ is any function.}$$

Solution:

Capillary rise (h) m $\equiv [L]$

Specific weight (sp.wt) N/m^3 $(MLT^{-2}\,L^{-3}) \equiv [ML^{-2}T^{-2}]$

Surface tension (σ) N/m $(MLT^{-2}\,L^{-1}) \equiv [MT^{-2}]$

Tube radius (r) m $\equiv [L]$

$$h = f\,(sp.wt., \sigma, r)$$

$$h = k\,(sp.wt.^a, \sigma^b, r^c)$$

$$[L] = [ML^{-2}T^{-2}]^a\,[MT^{-2}]^b[L]^c$$

Now by the principle of dimensional homogeneity, equating the power of M, L, T on both sides of the equation

For M $\quad 0 = a + b \quad \Rightarrow \quad a = -b$

For L $\quad 1 = -2a + c \quad \Rightarrow \quad a = -b$

For T $\quad 0 = -2a - 2b \quad \Rightarrow \quad a = -b$

$$h = k\,(sp.wt.^{-b}, \sigma^b, r^{1-2b})$$

$$h = k\,r\left(\frac{\sigma}{sp.wt.\,r^2}\right)^b \qquad \therefore\; h = r\phi\left(\frac{\sigma}{(sp.wt.)\,r^2}\right)$$

Example -2.3-

Prove that the resistance (F) of a sphere of diameter (d) moving at a constant speed (u) through a fluid density (ρ) and dynamic viscosity (μ) may be expressed as:

$$F = \frac{\mu^2}{\rho}\phi\left(\frac{\rho\, u\, d}{\mu}\right),$$ where ϕ is any function.

Solution:

Resistance (F) N	$\equiv [MLT^{-2}]$
Diameter (d) m	$\equiv [L]$
Speed (u) m/s	$\equiv [LT^{-1}]$
Density (ρ) kg/m^3	$\equiv [ML^{-3}]$
Viscosity (μ) kg/m.s	$\equiv [ML^{-1}\,T^{-1}]$

$$F = f\,(d, u, \rho, \mu)$$

$$F = k\,(d^a, u^b, \rho^c, \mu^d)$$

$$[MLT^{-2}] = [L]^a\,[LT^{-1}]^b[ML^{-3}]^c[ML^{-1}T^{-1}]^d$$

For M $\quad 1 = c + d \quad \Rightarrow \quad c = 1 - b$ ---------------(1)

For L $\quad 1 = a + b - 3c - d$ -----------------------(2)

For T $\quad -2 = -b - d \quad \Rightarrow \quad b = 2 - b$ ---------------(3)

By substituting equations (1) and (2) in equation (3) give

$$a = 1 - b + 3c + d = 1 - (2 - d) + 3\,(1 - d) + d = 2 - d$$

$$F = k\,(d^{2-d}, u^{2-d}, \rho^{1-d}, \mu^{d}) = k\,\{(d^2\,u^2\,\rho)\,(\mu / \rho\,u\,d)^{d}\} \text{ ---------x } \{(\rho/\,\mu^2) / (\rho/\,\mu^2)\}$$

$$F = k\,\{(d^2\,u^2\,\rho^2 / \mu^2)\,(\mu / \rho\,u\,d)^{d}\,(\mu^2 / \rho)\}$$

$$\therefore\; F = \frac{\mu^2}{\rho}\,\phi\left(\frac{\rho\,u\,d}{\mu}\right)$$

Example -2.4-

The thrust (P) (قوة الدفع) of a propeller depends upon diameter (D); speed (u) through a fluid density (ρ); revolution per minute (N); and dynamic viscosity (μ) Show that:

$$P = (\rho\,D^2\,u^2)\,f\left((\frac{\mu}{\rho\,D\,u}),(\frac{D\,N}{u})\right), \text{ where } f \text{ is any function.}$$

Solution:

Thrust (P) N	$\equiv [MLT^{-2}]$
Diameter (D) m	$\equiv [L]$
Speed (u) m/s	$\equiv [LT^{-1}]$
Density (ρ) kg/m^3	$\equiv [ML^{-3}]$
Rev. per min. (N) min^{-1}	$\equiv [T^{-1}]$
Viscosity (μ) kg/m.s	$\equiv [ML^{-1}\,T^{-1}]$

$$P = f\,(D, u, \rho, N, \mu)$$

$$P = k\,(D^a, u^b, \rho^c, N^d, \mu^e)$$

$$[MLT^{-2}] = [L]^a\,[LT^{-1}]^b[ML^{-3}]^c[T^{-1}][ML^{-1}T^{-1}]^d$$

For M $\quad 1 = c + e \quad \Rightarrow \quad c = 1 - e$ --------------(1)

For L $\quad 1 = a + b - 3c - e \quad \Rightarrow \quad a = 1 - b + 3c + e$ --------------(2)

For T $\quad -2 = -b - d - e \quad \Rightarrow \quad b = 2 - e - d$ --------------(3)

By substituting equations (1) and (3) in equation (2) give

$a = 1 - (2 - e - d) + 3\,(1 - e) + e = 2 - e + d$

$P = k\,(D^{2-e+d}, u^{2-e+d}, \rho^{1-e}, N^d, \mu^e)$

$$P = (\rho\, D^2\, u^2)\, k\left[\left(\frac{\mu}{\rho\, D\, u}\right)^e, \left(\frac{D\,N}{u}\right)\right]$$

$$\therefore\; P = (\rho\, D^2\, u^2)\, f\left((\frac{\mu}{\rho\, D\, u}), (\frac{D\,N}{u})\right)$$

Home Work

P.2.1

Show, by dimensional analysis, that the power (P) developed by a hydraulic turbine is given by; $P = (\rho \ N^3 \ D^5) \ f\left((\frac{N^2 \ D^2}{g \ H}) \right)$ where (ρ) is the fluid density, (N) is speed of rotation in r.p.m., (D) is the diameter of runner, (H) is the working head, and (g) is the gravitational acceleration.

P.2.2

The resistance (R) experienced by a partially submerged body depends upon the velocity (u), length of the body (L), dynamic viscosity (μ) and density (ρ) of the fluid, and gravitational acceleration (g). Obtain a dimensionless expression for (R).

Ans. $R = (u^2 L^2 \rho) \ f\left((\frac{\mu}{u \ L \ g}), (\frac{L \ g}{u^2}) \right)$

P.2.3

Using Rayleigh's method to determine the rational formula for discharge (Q) through a sharp-edged orifice freely into the atmosphere in terms of head (h), diameter (d), density (ρ), dynamic viscosity (μ), and gravitational acceleration (g).

Ans. $Q = (d \sqrt{g \ h}) f\left[(\frac{\mu}{\rho \ d^{\frac{3}{2}} \ g^{\frac{1}{2}}}), (\frac{h}{d}) \right]$

2.4.2 Buckingham's method (or Π-Theorem)

It has been observed that the Rayleigh's method of dimensional analysis becomes cumbersome, when a large number of variables are involved. In order to overcome this difficulty, the Buckingham's method may be convenient used. It states that " If there are **(n)** variables in a dimensionally homogeneous equation, and if these variables contain **(m)** fundamental dimensions such as (MLT) they may be grouped into **(n-m)** non-dimensional independent Π-terms".

Mathematically, if a dependent variable X_1 depends upon independent variables ($X_2, X_3, X_4, \ldots\ldots\ldots. X_n$), the functional equation may be written as:

$$X_1 = k\,(X_2, X_3, X_4, \ldots\ldots\ldots. X_n)$$

This equation may be written in its general form as;

$$f\,(X_1, X_2, X_3, \ldots\ldots\ldots. X_n) = 0$$

In this equation, there are n variables. If there are m fundamental dimensions, then according to Buckingham's Π-theorem;

$$f_1\,(\Pi_1, \Pi_2, \Pi_3, \ldots\ldots\ldots. \Pi_{n-m}) = 0$$

The Buckingham's Π-theorem is based on the following steps:

1. First of all, write the functional relationship with the given data.
2. Then write the equation in its general form.
3. Now choose **m** *repeating variables (or recurring set)* and write separate expressions for each Π-term. Every Π-term will contain the repeating variables and one of the remaining variables. **Just** the repeating variables are written in exponential form.
4. With help of the principle of dimensional homogeneity find out the values of powers a, b, c, …… by obtaining simultaneous equations.
5. Now substitute the values of these exponents in the Π-terms.
6. After the Π-terms are determined, write the functional relation in the required form.

2.4.2.1 Selection of repeating variables

In the previous section, we have mentioned that we should choose **(m)** repeating variables and write separate expressions for each Π-term. Though there is no hard or fast rule for the selection of repeating variables, yet the following points should be borne in mind while selecting the repeating variables:

1. The variables should be such that none of them is dimension les.
2. No two variables should have the same dimensions.
3. Independent variables should, as far as possible, be selected as repeating variables.
4. Each of the fundamental dimensions must appear in at least one of the **m** variables.
5. It must not possible to form a dimensionless group from some or all the variables within the repeating variables. If it were so possible, this dimensionless group would, of course, be one of the Π-term.
6. In general the selected repeating variables should be expressed as the following: **(1)** representing the flow characteristics, **(2)**, representing the geometry and **(3)** representing the physical properties of fluid.
7. In case of that the example is held up, then one of the repeating variables should be changed.

Example -2.5-

By dimensional analysis, obtain an expression for the drag force (F) on a partially submerged body moving with a relative velocity (u) in a fluid; the other variables being the linear dimension (L), surface roughness (e), fluid density (ρ), and gravitational acceleration (g).

Solution:

Drag force (F) N	$\equiv [MLT^{-2}]$
Relative velocity (u) m/s	$\equiv [LT^{-1}]$
Linear dimension (L) m	$\equiv [L]$
Surface roughness (e) m	$\equiv [L]$
Density (ρ) kg/m^3	$\equiv [ML^{-3}]$
Acceleration of gravity (g) m/s^2	$\equiv [ML^{-1}\,T^{-1}]$

$$F = k\,(u, L, e, \rho, g)$$

$$f\,(F, u, L, e, \rho, g) = 0$$

$$n = 6,\ m = 3,\ \Rightarrow \Pi = n - m = 6 - 3 = 3$$

No. of repeating variables = m = 3

The selected repeating variables is (u, L, ρ)

$$\Pi_1 = u^{a1}\,L^{b1}\,\rho^{c1}\,F \qquad \text{--------------(1)}$$

$$\Pi_2 = u^{a2}\,L^{b2}\,\rho^{c2}\,e \qquad \text{--------------(2)}$$

$$\Pi_3 = u^{a3}\,L^{b3}\,\rho^{c3}\,g \qquad \text{--------------(3)}$$

For Π_1 equation (1)

$$[M^0\,L^0\,T^0] = [L\,T^{-1}]^{a1}\,[L]^{b1}[ML^{-3}]^{c1}[MLT^{-2}]$$

Now applied dimensional homogeneity

For M	$0 = c1 + 1$	$\Rightarrow$	$c1 = -1$
For T	$0 = -a1 - 2$	$\Rightarrow$	$a1 = -2$
For L	$0 = a1 + b1 - 3c1 + 1$	$\Rightarrow$	$b1 = -2$

$$\Pi_1 = u^{-2} L^{-2} \rho^{-1} F \qquad \Rightarrow \quad \Pi_1 = \frac{F}{u^2 \, L^2 \, \rho}$$

For Π_2 equation (2)

$$[M^0 L^0 T^0] = [L\, T^{-1}]^{a2} [L]^{b2} [ML^{-3}]^{c2} [L]$$

For M	$0 = c2$	$\Rightarrow$	$c2 = 0$
For T	$0 = -a2$	$\Rightarrow$	$a2 = 0$
For L	$0 = a2 + b2 - 3c2 + 1$	$\Rightarrow$	$b2 = -1$

$$\Pi_2 = L^{-1} e \qquad \Rightarrow \quad \Pi_2 = \frac{e}{L}$$

For Π_3 equation (3)

$$[M^0 L^0 T^0] = [L T^{-1}]^{a3} [L]^{b3} [ML^{-3}]^{c3} [L T^{-2}]$$

For M $\quad 0 = c3 \quad \Rightarrow \quad c3 = 0$

For T $\quad 0 = -a3 - 2 \quad \Rightarrow \quad a3 = -2$

For L $\quad 0 = a3 + b3 - 3c3 + 1 \quad \Rightarrow \quad b3 = 1$

$$\Pi_3 = u^{-2} L g \quad \Rightarrow \quad \Pi_3 = \frac{L g}{u^2}$$

$$f_1(\Pi_1, \Pi_2, \Pi_3) = 0 \quad \Rightarrow \quad f_1\left(\frac{F}{u^2 L^2 \rho}, \frac{e}{L}, \frac{L g}{u^2}\right) = 0$$

$$\therefore \ F = u^2 L^2 \rho \, f\left(\frac{e}{L}, \frac{L g}{u^2}\right)$$

Example -2.6-

Prove that the discharge (Q) over a spillway (قناة لتصريف فائض المياه من سد او نهر) is given by the relation $Q = u\,D^2\, f(\frac{\sqrt{g\,D}}{u}, \frac{H}{D})$ where (u) velocity of flow (D) depth at the throat, (H), head of water, and (g) gravitational acceleration.

Solution:

Discharge (Q) m^3/s	$\equiv [L^3T^{-1}]$
Velocity (u) m/s	$\equiv [LT^{-1}]$
Depth (D) m	$\equiv [L]$
Head of water (H) m	$\equiv [L]$
Acceleration of gravity (g) m/s^2	$\equiv [ML^{-1}\,T^{-1}]$

$$Q = k\,(u, D, H, g)$$

$$f\,(Q, u, D, H, g) = 0$$

$$n = 5,\ m = 2,\ \Rightarrow \Pi = n - m = 5 - 2 = 3$$

No. of repeating variables = m = 2

The selected repeating variables is (u, D)

$$\Pi_1 = u^{a1}\,D^{b1}\,Q \quad \text{--------------(1)}$$

$$\Pi_2 = u^{a2}\,D^{b2}\,H \quad \text{--------------(2)}$$

$$\Pi_3 = u^{a3}\,D^{b3}\,g \quad \text{--------------(3)}$$

For Π_1 equation (1)

$$[M^0 L^0 T^0] = [L T^{-1}]^{a1} [L]^{b1} [L^3 T^{-1}]$$

For T	$0 = -a1 - 1$	$\Rightarrow$	$a1 = -1$
For L	$0 = a1 + b1 + 3$	$\Rightarrow$	$b1 = -2$

$$\Pi_1 = u^{-1} D^{-2} Q \quad \Rightarrow \quad \Pi_1 = \frac{Q}{u D^2}$$

For Π_2 equation (2)

$$[M^0 L^0 T^0] = [L T^{-1}]^{a2} [L]^{b2} [L]$$

For T	$0 = -a2$	$\Rightarrow$	$a2 = 0$
For L	$0 = a2 + b2 + 1$	$\Rightarrow$	$b2 = -1$

$$\Pi_2 = D^{-1} H \quad \Rightarrow \quad \Pi_2 = \frac{D}{H}$$

For Π_2 equation (2)

$$[M^0\, L^0\, T^0] = [L\, T^{-1}]^{a2}\, [L]^{b2} [L]$$

For T $\quad 0 = -\, a2 \quad \Rightarrow \quad a2 = 0$

For L $\quad 0 = a2 + b2 + 1 \quad \Rightarrow \quad b2 = -\, 1$

$$\Pi_2 = D^{-1}\, H \qquad \Rightarrow \quad \Pi_2 = \frac{D}{H}$$

For Π_3 equation (3)

$$[M^0\, L^0\, T^0] = [L\, T^{-1}]^{a3}\, [L]^{b3}\, [L\, T^{-2}]$$

For T $\quad 0 = -\, a3 - 2 \quad \Rightarrow \quad a3 = -\, 2$

For L $\quad 0 = a3 + b3 + 1 \quad \Rightarrow \quad b3 = 1$

$$\Pi_3 = u^{-2}\, D\, g \qquad \Rightarrow \quad \Pi_3 = \frac{D\, g}{u^2} = \frac{\sqrt{g\, D}}{u}$$

$$f_1\, (\Pi_1, \Pi_2, \Pi_3) = 0 \qquad \Rightarrow \quad f_1\left(\frac{Q}{u\, D^2}, \frac{D}{H}, \frac{\sqrt{D\, g}}{u}\right)$$

$$\therefore\; Q = u\, D^2\, f\left(\frac{\sqrt{g\, D}}{u}, \frac{H}{D}\right)$$