INC 341 Feedback Control Systems: Lecture 3 Transfer Function of Dynamic Systems II

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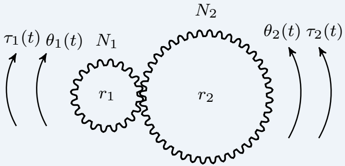
Transfer Functions for Systems with Gears

- Gears provide mechanical advantage to rotational system, e.g. a bicycle with gears.
- Gears are nonlinear. They exhibit *backlash*, which occurs from the loose fit between two meshed gears.
- In this course, we consider only the linearized version of gears.

A gear system



- a small gear has radius r_1 and N_1 teeth is rotated through angle $\theta_1(t)$ due to a torque, $\tau_1(t)$.
- a big gear have radius r_2 and N_2 teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $\tau_2(t)$.

Transfer Functions for Systems with Gears

- The gears turn, the distance traveled along each gear's circumference is the same. Thus

$$r_1\theta_1 = r_2\theta_2$$

or

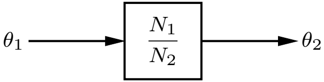
$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

- If the gears are lossless, that is they do not absorb or store energy, the energy into Gear 1 equals the energy out of Gear 2. Since the translational energy of force times displacement becomes the rotational energy of torque time angular displacement.

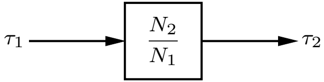
$$\tau_1\theta_1 = \tau_2\theta_2$$

or

$$\frac{\tau_2}{\tau_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

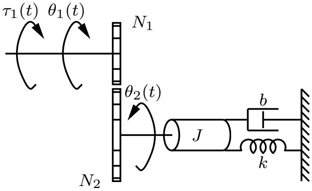


(a)



(b)

- (a) is a transfer functions for angular displacement in lossless gears
- (b) is a transfer functions for torque in lossless gears.



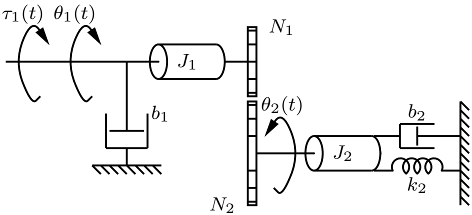
The first gear (lossless) generates torque (τ_1) to drive the second gear by τ_2, then

$$\left(Js^2 + bs + k\right)\Theta_2(s) = \hat{\tau}_2(s)$$

$$\frac{\tau_2}{\tau_1} = \frac{N_2}{N_1} = \frac{\theta_1}{\theta_2}$$

$$\left(Js^2 + bs + k\right)\frac{N_1}{N_2}\Theta_1(s) = \frac{N_2}{N_1}\hat{\tau}_1(s)$$

$$\left(J\left(\frac{N_1}{N_2}\right)^2 s^2 + b\left(\frac{N_1}{N_2}\right)^2 s + k\left(\frac{N_1}{N_2}\right)^2\right)\Theta_1(s) = \hat{\tau}_1(s)$$



Find the transfer function, $\Theta_2(s)/\hat{\tau}_1(s)$. Assuming that $\tau_e(t)$ is the torque generates at the first gear by the torque $\tau_1(t)$, the we have

$$\left(J_1 s^2 + b_1 s\right)\Theta_1(s) + \hat{\tau}_e(s) = \hat{\tau}_1(s)$$

$$\left(J_1 s^2 + b_1 s\right)\frac{N_2}{N_1}\Theta_2(s) + \hat{\tau}_e(s) = \hat{\tau}_1(s)$$

Transfer Functions for Systems with Gears

Transfer functions for lossless gears

At the second gear, we have

$$\left(J_2 s^2 + b_2 s + k_2\right) \Theta_2(s) = \hat{\tau}_2(s)$$

$$\left(J_2 s^2 + b_2 s + k_2\right) \Theta_2(s) = \frac{N_2}{N_1}\hat{\tau}_e(s)$$

$$\left(J_2 s^2 + b_2 s + k_2\right) \frac{N_1}{N_2}\Theta_2(s) = \hat{\tau}_e(s)$$

Combining the equations of both gears, we have

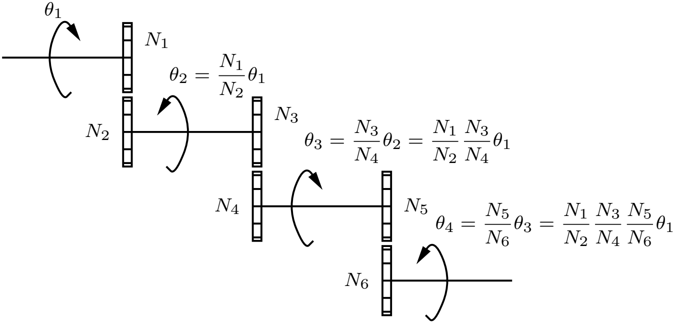
$$\left[J_1\left(\frac{N_2}{N_1}\right)s^2 + b_1\left(\frac{N_2}{N_1}\right)s + J_2\left(\frac{N_1}{N_2}\right)s^2 + b_2\left(\frac{N_1}{N_2}\right)s + k_2\left(\frac{N_1}{N_2}\right)\right]\Theta_2(s) = \hat{\tau}_1(s)$$

$$\left[\left(J_1\left(\frac{N_2}{N_1}\right)^2 + J_2\right)s^2 + \left(b_1\left(\frac{N_2}{N_1}\right)^2 + b_2\right)s + k_2\right]\frac{N_1}{N_2}\Theta_2(s) = \hat{\tau}_1(s)$$

$$\frac{\Theta_2(s)}{\hat{\tau}_2(s)} = \frac{N_2/N_1}{\left[\left(J_1\left(\frac{N_2}{N_1}\right)^2 + J_2\right)s^2 + \left(b_1\left(\frac{N_2}{N_1}\right)^2 + b_2\right)s + k_2\right]}$$

Transfer Functions for Systems with Gears

Transfer functions for lossless gears

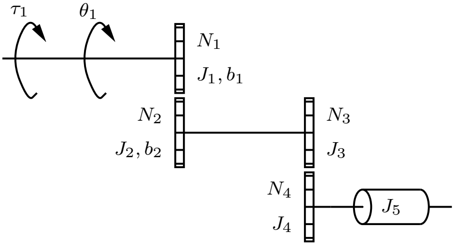


A gear train can use instead of one large radii gear. This can be done by cascading smaller gear ratios. As shown in above figure,

$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6}\theta_1$$

Transfer Functions for Systems with Gears

Transfer functions for lossless gears



Starting from the left most of the gear, we can find the transfer function $\Theta_1(s)/\hat{\tau}_1(s)$ as follow:

$$\left(J_1 s^2 + b_1 s\right)\Theta_1(s) + \hat{\tau}_{e1}(s) = \hat{\tau}_1(s)$$
$$\left[\left(J_2 + J_3\right)s^2 + b_2 s\right]\Theta_2(s) + \hat{\tau}_{e2}(s) = \hat{\tau}_2(s)$$
$$\left[\left(J_4 + J_5\right)s^2\right]\Theta_4(s) = \hat{\tau}_4(s)$$

Transform all torques and angle to be in terms of $\hat{\tau}_1(s)$ and $\Theta_1(s)$ respectively.

$$\left[(J_4+J_5)\,s^2\right]\frac{N_3}{N_4}\Theta_2(s)=\frac{N_4}{N_3}\hat{\tau}_{e2}(s)$$

Substituting $\hat{\tau}_{e2}(s)$ to one above equation, we have

$$\left[(J_2+J_3)\,s^2+(J_4+J_5)\left(\frac{N_3}{N_4}\right)^2s^2+b_2s\right]\Theta_2(s)=\hat{\tau}_2(s)$$

$$\left[(J_2+J_3)\,s^2+(J_4+J_5)\left(\frac{N_3}{N_4}\right)^2s^2+b_2s\right]\frac{N_1}{N_2}\Theta_1(s)=\frac{N_2}{N_1}\hat{\tau}_{e1}(s)$$

Substituting $\hat{\tau}_{e1}(s)$ to one above equation, we have

$$\left[\left(J_1+(J_2+J_3)\left(\frac{N_1}{N_2}\right)^2+(J_4+J_5)\left(\frac{N_1}{N_2}\frac{N_3}{N_4}\right)^2\right)s^2+\left(b_1+b_2\left(\frac{N_1}{N_2}\right)^2\right)s\right]\Theta_1(s)=\hat{\tau}_1(s)$$

$$\frac{\Theta_1(s)}{\hat{\tau}_1(s)}=\frac{1}{\left[\left(J_1+(J_2+J_3)\left(\frac{N_1}{N_2}\right)^2+(J_4+J_5)\left(\frac{N_1}{N_2}\frac{N_3}{N_4}\right)^2\right)s^2+\left(b_1+b_2\left(\frac{N_1}{N_2}\right)^2\right)s\right]}$$

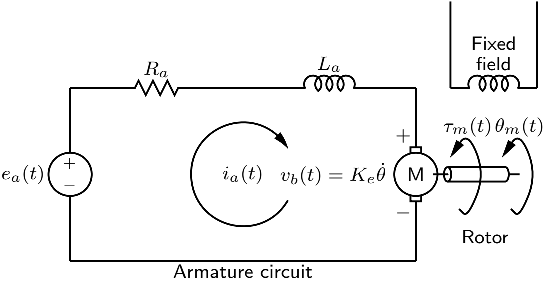
Electromechanical System Transfer Functions

An *electromechanical systems* is a hybrid system of electrical and mechanical variables. This system has a lot of application for examples

- an antenna azimuth position control system
- robot and robot arm controls
- sun and star trackers
- disk-drive position controls

Industrial robot arm

Electromechanical System Transfer Functions



- $v_b(t) = K_b \dfrac{d\theta_m(t)}{dt}$, where $v_b(t)$ is the *back electromotive force (back emf)*; K_b is a constant of proportionality called the back emf constant.
- The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, $v_b(t)$ is

$$R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t) = e_a(t)$$

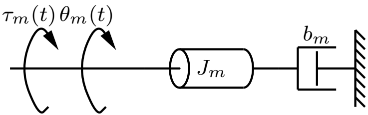
Electromechanical System Transfer Functions

- The torque developed by the motor is proportional to the armature current; thus

$$\tau_m(t) = K_t i_a(t),$$

  where $\tau_m(t)$ is the torque developed by the motor, and K_t is a constant for proportionality, called the motor torque constant.
- Taking the Laplace transform of both relationship and substituting $I_a(s)$ into the mesh equation, we have

$$\frac{(R_a + L_a s)\hat{\tau}_m(s)}{K_t} + K_b s \Theta_m(s) = E_a(s)$$



The figure shows a typical equivalent mechanical loading on a motor. We have

$$\hat{\tau}_m(s) = \left(J_m s^2 + b_m s\right) \Theta_m(s)$$

Electromechanical System Transfer Functions

Substituting $\hat{\tau}_m(s)$ into the armature equation yields

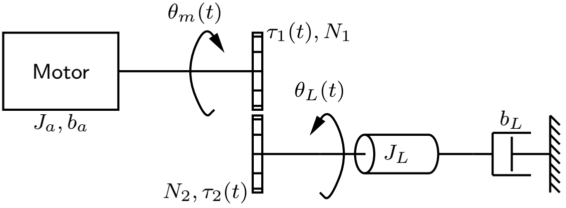
$$\frac{(R_a + L_a s)\left(J_m s^2 + b_m s\right)\Theta_m(s)}{K_t} + K_b s\Theta_m(s) = E_a(s)$$

Assuming that the armature inductance, L_a is small compared to the armature resistance, R_a, the equation become

$$\left[\frac{R_a}{K_t}(J_m s + b_m) + K_b\right] s\Theta_m(s) = E_a(s)$$

After simplification, the desired transfer function is

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(b_m + \frac{K_t K_b}{R_a}\right)\right]}$$



A motor with inertia J_a and damping b_a at the armature driving a load consisting of inertia J_L and damping b_L. Assuming that J_a, J_L, b_a, and b_L are known. Then, we have

$$\left(J_a s^2 + b_a s\right) \Theta_m(s) + \hat{\tau}_1(s) = 0.$$

At the load side, we obtain

$$\left(J_L s^2 + b_L s\right) \Theta_L(s) = \hat{\tau}_2(s)$$
$$\left(J_L s^2 + b_L s\right) \frac{N_1}{N_2} \Theta_m(s) = \frac{N_2}{N_1} \hat{\tau}_1(s)$$

Substituting the $\hat{\tau}_1(s)$ back to the motor side equation, the equivalent equation is

$$\left[\left(J_a + J_L\left(\frac{N_1}{N_2}\right)^2\right)s^2 + \left(b_a + b_L\left(\frac{N_1}{N_2}\right)^2\right)s\right]\Theta_m(s) = 0$$

Or the equivalent inertial, J_m, and the equivalent damping, b_m, at the armature are

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2; \qquad b_m = b_a + b_L\left(\frac{N_1}{N_2}\right)^2$$

Next step, we going to find the electrical constants by using a *dynamometer* test of motor. This can be done by measuring the torque and speed of a motor under the condition of a constant applied voltage. Substituting $V_b(s) = K_b s\Theta_m(s)$ and $\hat{\tau}_m(s) = K_t I_a(s)$ in to the Laplace transformed armature circuit, with $L_a = 0$, yields

$$\frac{R_a}{K_t}\hat{\tau}_m(s) + K_b s\Theta_m(s) = E_a(s)$$

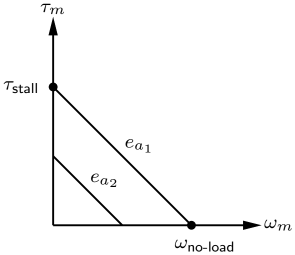
Taking the inverse Laplace transform, we get

$$\frac{R_a}{K_t}\tau_m(t) + K_b\omega_m(t) = e_a(t)$$

If $e_a(t)$ is a DC voltage, at the steady state, the motor should turn a a constant speed, ω_m, with a constant torque, τ_m. With this, we have

$$\frac{R_a}{K_t}\tau_m + K_b\omega_m = e_a \qquad \Rightarrow \qquad \tau_m = -\frac{K_tK_b}{R_a}\omega_m + \frac{K_t}{R_a}e_a$$

Torque-peed curve



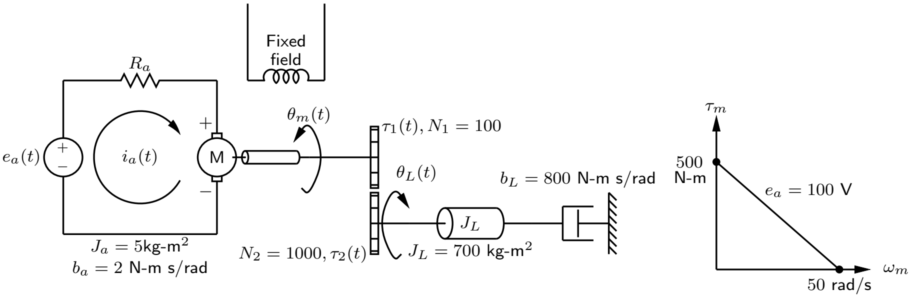
- $\omega_m = 0$, the value of torque is called the *stall torque*, τ_{stall}. Thus

$$\tau_{\text{stall}} = \frac{K_t}{R_a}e_a \Rightarrow \frac{K_t}{R_a} = \frac{\tau_{\text{stall}}}{e_a}$$

- $\tau_m = 0$, the angular velocity becomes *no-load speed*, $\omega_{\text{no-load}}$. Thus

$$\omega_{\text{no-load}} = \frac{e_a}{K_b} \Rightarrow K_b = \frac{e_a}{\omega_{\text{no-load}}}$$

Transfer Function–DC Motor and Load



Find $\Theta_L(s)/E_a(s)$ from the given system and torque-speed curve. The total inertia and the total damping at the armature of the motor are

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12$$

$$b_m = b_a + b_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10$$

Transfer Function–DC Motor and Load

Next, we find the electrical constants K_t/R_a and K_b from the torque-speed curve. Hence,

$$\frac{K_t}{R_a} = \frac{\tau_{\text{stall}}}{e_a} = \frac{500}{100} = 5$$

and

$$K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{100}{50} = 2$$

We have

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(b_m + \frac{K_t K_b}{R_a}\right)\right]} = \frac{0.417}{s(s+1.667)}.$$

Using the gear ratio, $N_1/N_2 = 0.1$

$$\frac{\Theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)}$$

Reference

1. Norman S. Nise, "*Control Systems Engineering*,6^{th} edition, Wiley, 2011
2. Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini, "*Feedback Control of Dyanmic Systems*",4^{th} edition, Prentice Hall, 2002