CHAPTER#8

8. Time Domain Analysis of Control Systems

8.1. Time response

Time response c(t)is the variation of output with respect to time. The part of time response that goes to zero after large interval of time is called transient response $c_{tr}(t)$. The part of time response that remains after transient response is called steady-state response $c_{ss}(t)$.

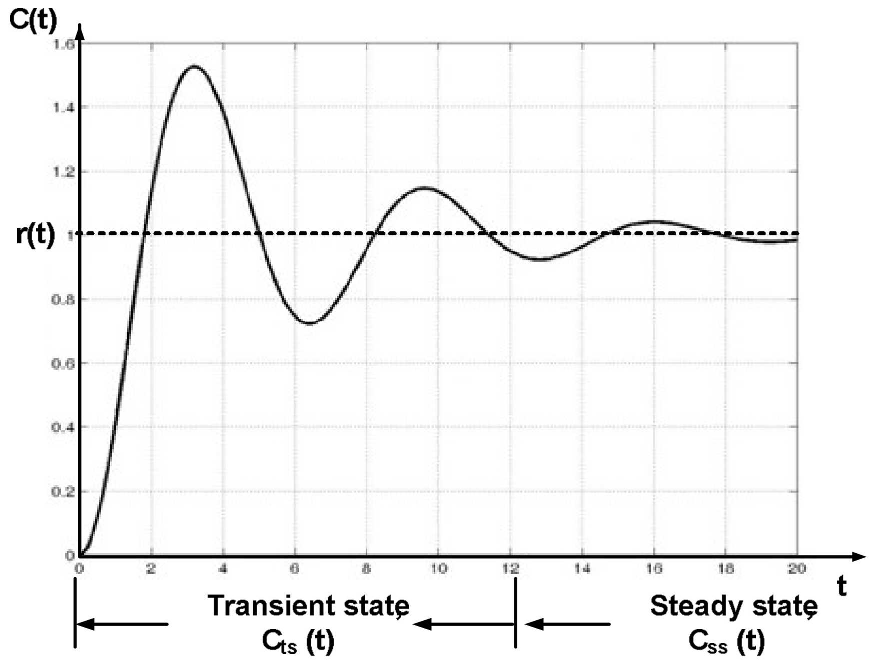


Fig.7.1. Time response of a system

8.2. System dynamics

System dynamics is the study of characteristic and behaviour of dynamic systems

i.e.

i. Differential equations: First-order systems and Second-order systems,
ii. Laplace transforms,
iii. System transfer function,
iv. Transient response: Unit impulse, Step and Ramp

Laplace transforms convert differential equations into algebraic equations. They are related to frequency response

$$\mathbf{L}\{\mathbf{x}(\mathbf{t})\} = \mathbf{X}(\mathbf{s}) = \int_0^\infty x(t)e^{-st}\,dt \tag{8.1}$$

No.	Function	Time-domain $x(t)= \mathcal{L}^{-1}\{X(s)\}$	Laplace domain $X(s)= \mathcal{L}\{x(t)\}$
1	Delay	$\delta(t-\tau)$	$e^{-\tau s}$
2	Unit impulse	$\delta(t)$	1
3	Unit step	u(t)	$\frac{1}{s}$
4	Ramp	t	$\frac{1}{s^2}$
5	Exponential decay	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
6	Exponential approach	$\left(1-e^{-\alpha t}\right)$	$\frac{\alpha}{s(s+\alpha)}$
7	Sine	sin ωt	$\frac{\omega}{s^2+\omega^2}$
8	Cosine	cos ωt	$\frac{s}{s^2+\omega^2}$
9	Hyperbolic sine	sinh αt	$\frac{\alpha}{s^2-\alpha^2}$
10	Hyperbolic cosine	cosh αt	$\frac{s}{s^2-\alpha^2}$
11	Exponentially decaying sine wave	$e^{-\alpha t}\sin\omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
12	Exponentially decaying cosine wave	$e^{-\alpha t}\cos\omega t$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

8.3. Forced response

$$C(s) = G(s)R(s) = \frac{K(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}R(s) \tag{8.2}$$

R(s) input excitation

8.4. Standard test signals

8.4.1. Impulse Signal: An impulse signal δ(t) is mathematically defined as follows.

$$\delta(t) = \left.\begin{matrix} undefined & ;t=0 \\ 0 & ;t\neq 0 \end{matrix}\right\} \tag{8.3}$$

Laplace transform of impulse signal is

$$\delta(s) = 1 \tag{8.4}$$

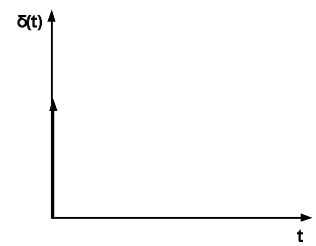
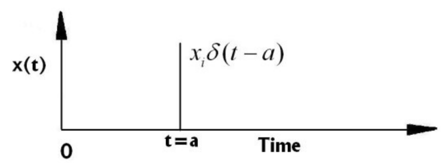


Fig.7.2. Impulse signal

Dirac delta function

$$x(t) = x_I\delta(t-a) \tag{8.5}$$



Integral property of Dirac delta function

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-t_o)\,dt = \phi(t_o) \tag{8.6}$$

Laplace transform of an impulse input

$$X(s) = \int_{0}^{\infty} e^{-st} x_I\delta(t-a)\,dt = x_I e^{-sa} \tag{8.7}$$

8.4.2. Step Signal: A step signal u(t) is mathematically defined as follows.

$$u(t) = \left.\begin{matrix} 0 & ;t<0 \\ K & ;t\geq 0 \end{matrix}\right\} \tag{8.8}$$

Laplace transform of step signal is

$$U(s) = \frac{K}{s} \tag{8.9}$$

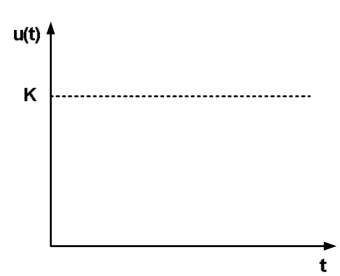


Fig.7.2. Step signal

8.4.3. Ramp Signal: A step signal r(t) is mathematically defined as follows.

$$r(t) = \left.\begin{matrix} 0 & ; t < 0 \\ Kt & ; t \geq 0 \end{matrix}\right\} \quad (8.10)$$

Laplace transform of ramp signal is

$$R(s) = \frac{K}{s^2} \quad (8.11)$$

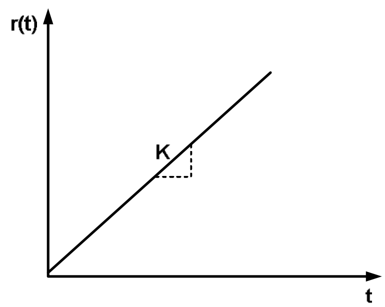


Fig.7.3. Ramp signal

8.4.4. Parabolic Signal A step signal a(t) is mathematically defined as follows.

$$a(t) = \left.\begin{matrix} 0 & ; t < 0 \\ \frac{Kt^2}{2} & ; t \geq 0 \end{matrix}\right\} \quad (8.12)$$

Laplace transform of parabolic signal is

$$A(s) = \frac{K}{s^3} \quad (8.13)$$

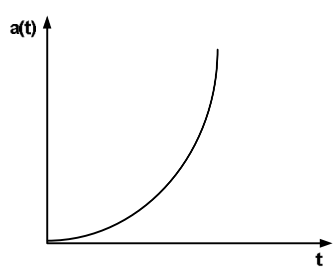


Fig.7.4. Parabolic signal

8.4.5. Sinusoidal Signal A sinusoidal x(t) is mathematically defined as follows.

$$x(t) = \sin \omega t \tag{8.14}$$

Laplace transform of sinusoidal signal is

$$X(s) = \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{\omega}{s^2 + \omega^2} \tag{8.15}$$

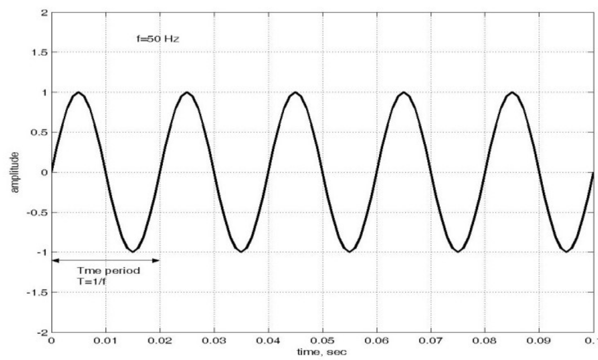


Fig.7.4. Sinusoidal signal

8.5. Steady-state error:

A simple closed-loop control system with negative feedback is shown as follows.

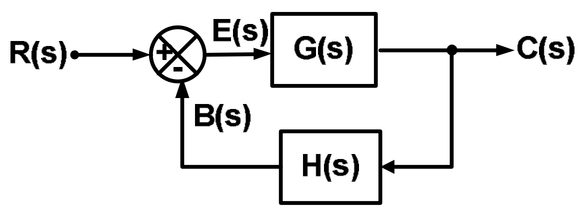


Fig.7.5. A simple closed-loop control system with negative feedback

Here,

$$E(s) = R(s) - B(s) \tag{8.16}$$

$$B(s) = C(s)H(s) \tag{8.17}$$

$$C(s) = E(s)G(s) \tag{8.18}$$

Applying (1) in (9),

$$E(s) = R(s) - C(s)H(s) \tag{8.19}$$

Using (11) in (12),

$$E(s) = R(s) - E(s)G(s)H(s) \tag{8.20}$$

$$\Rightarrow \left[1 + G(s)H(s)\right]E(s) = R(s) \tag{8.21}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)H(s)} \tag{8.22}$$

Steady-state error,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \tag{8.23}$$

Using (15) in (16),

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} \tag{8.24}$$

Therefore, steady-state error depends on two factors, i.e.

(a) type and magnitude of R(s)
(b) open-loop transfer function G(s)H(s)

8.6. Types of input and Steady-state error:

8.6.1. Step Input

$$R(s) = \frac{A}{s} \tag{8.25}$$

Using (18) in (17),

$$e_{ss} = \lim_{s\to 0} \frac{s\left(\frac{A}{s}\right)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{A}{1+G(s)H(s)} \tag{8.26}$$

$$\Rightarrow e_{ss} = \frac{A}{1+\lim_{s\to 0} G(s)H(s)} = \frac{A}{1+K_P} \tag{8.27}$$

Where,

$$K_P = \lim_{s\to 0} G(s)H(s) \tag{8.28}$$

8.6.2. Ramp Input

$$R(s) = \frac{A}{s^2} \tag{8.29}$$

Using (18) in (17),

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} \frac{s\left(\frac{A}{s^2}\right)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{A}{s\left[1+G(s)H(s)\right]} \\ \Rightarrow e_{ss} &= \lim_{s\to 0} \frac{A}{s+sG(s)H(s)} \\ \Rightarrow e_{ss} &= \frac{A}{\lim_{s\to 0} sG(s)H(s)} = \frac{A}{K_V} \end{aligned} \tag{8.30}$$

Where,

$$K_V = \lim_{s\to 0} sG(s)H(s) \tag{8.31}$$

8.6.3. Parabolic Input

$$R(s) = \frac{A}{s^3} \tag{8.32}$$

Using (18) in (17),

$$e_{ss} = \lim_{s\to 0} \frac{s\left(\frac{A}{s^3}\right)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{A}{s^2\left[1+G(s)H(s)\right]}$$

$$\Rightarrow e_{ss} = \lim_{s\to 0} \frac{A}{s^2 + s^2G(s)H(s)} \tag{8.33}$$

$$\Rightarrow e_{ss} = \frac{A}{\lim\limits_{s\to 0} s^2G(s)H(s)} = \frac{A}{K_A}$$

Where,

$$K_A = \lim_{s\to 0} s^2G(s)H(s) \tag{8.34}$$

Types of input and steady-state error are summarized as follows.

Error Constant	**Equation**	**Steady-state error (e_{ss})**
Position Error Constant (K_P)	$K_P = \lim_{s\to 0} G(s)H(s)$	$e_{ss} = \frac{A}{1+K_P}$
Velocity Error Constant (K_V)	$K_V = \lim_{s\to 0} sG(s)H(s)$	$e_{ss} = \frac{A}{K_V}$
Acceleration Error Constant (K_A)	$K_A = \lim_{s\to 0} s^2G(s)H(s)$	$e_{ss} = \frac{A}{K_A}$

8.7. Types of open-loop transfer function G(s)H(s)and Steady-state error:

8.7.1. Static Error coefficient Method

The general form of G(s)H(s) is

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)\ldots(1+T_ns)}{s^j(1+T_as)(1+T_bs)\ldots(1+T_ms)} \tag{8.35}$$

Here, j = no. of poles at origin (s = 0)

or, type of the system given by eq (28) is j.

8.7.1.1. Type 0

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)\ldots(1+T_ns)}{(1+T_as)(1+T_bs)\ldots(1+T_ms)} \tag{8.36}$$

Here,

$$K_P = \lim_{s \to 0} G(s)H(s) = K \tag{8.37}$$

Therefore,

$$e_{ss} = \frac{A}{1+K} \tag{8.38}$$

8.7.1.2. Type 1

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots(1+T_n s)}{s(1+T_a s)(1+T_b s)\ldots(1+T_m s)} \tag{8.39}$$

Here,

$$K_V = \lim_{s \to 0} sG(s)H(s) = K \tag{8.40}$$

Therefore,

$$e_{ss} = \frac{A}{K} \tag{8.41}$$

8.7.1.3. Type 2

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots(1+T_n s)}{s^2(1+T_a s)(1+T_b s)\ldots(1+T_m s)} \tag{8.42}$$

Here,

$$K_A = \lim_{s \to 0} s^2 G(s)H(s) = K \tag{8.43}$$

Therefore,

$$e_{ss} = \frac{A}{K} \tag{8.44}$$

Steady-state error and error constant for different types of input are summarized as follows.

Type	**Step input**		**Ramp input**		**Parabolic input**	
	K_P	e_{ss}	K_V	e_{ss}	K_A	e_{ss}
Type 0	K	$\frac{A}{1+K}$	0	∞	0	∞
Type 1	∞	0	K	$\frac{A}{K}$	0	∞
Type 2	∞	0	∞	0	K	$\frac{A}{K}$

The static error coefficient method has following advantages:

- Can provide time variation of error
- Simple calculation

But, the static error coefficient method has following demerits:

- Applicable only to stable system
- Applicable only to three standard input signals
- Cannot give exact value of error. It gives only mathematical value i.e. 0 or ∞

8.7.2. Generalized Error coefficient Method

From eq (15),

$$E(s)=\left[\frac{1}{1+G(s)H(s)}\right]R(s)$$

So,

$$E(s)=F_1(s)F_2(s) \tag{8.45}$$

Where, $F_1=\dfrac{1}{1+G(s)H(s)}$ and $F_2(s)=R(s)$

Using convolution integral to eq (38)

$$e(t)=\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau=\int_0^t f_1(\tau)r(t-\tau)\,d\tau \tag{8.46}$$

Using Taylor's series of expansion to $r(t-\tau)$,

$$r(t-\tau)=r(t)-\tau r'(t)+\frac{\tau^2}{2!}r''(t)-\frac{\tau^3}{3!}r'''(t)+\ldots \tag{8.47}$$

Now, applying eq (40) in eq (39),

$$e(t)=\int_0^t f_1(\tau)r(t)\,d\tau-\int_0^t \tau r'(t)\,f_1(\tau)d\tau+\int_0^t \frac{\tau^2}{2!}r''(t)\,f_1(\tau)d\tau-\int_0^t \frac{\tau^3}{3!}r'''(t)\,f_1(\tau)d\tau+\ldots \tag{8.48}$$

Now, steady-state error, e_{ss} is

$$e_{ss}=\lim_{t\to\infty}e(t) \tag{8.49}$$

Therefore,

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{t\to\infty}\left[\int_0^t f_1(\tau)r(t)\,d\tau-\int_0^t \tau r'(t)\,f_1(\tau)d\tau+\int_0^t \frac{\tau^2}{2!}r''(t)\,f_1(\tau)d\tau-\int_0^t \frac{\tau^3}{3!}r'''(t)\,f_1(\tau)d\tau+\ldots\right]$$

$$\Rightarrow e_{ss}=\int_0^\infty f_1(\tau)r(t)\,d\tau-\int_0^\infty \tau r'(t)\,f_1(\tau)d\tau+\int_0^\infty \frac{\tau^2}{2!}r''(t)\,f_1(\tau)d\tau-\int_0^\infty \frac{\tau^3}{3!}r'''(t)\,f_1(\tau)d\tau+\ldots$$

(8.50)

Eq (44) can be rewritten as

$$e_{ss}=C_0r(t)+C_1r'(t)+\frac{C_2}{2!}r''(t)+\frac{C_3}{3!}r'''(t)+\ldots \tag{8.51}$$

Where, C_0, C_1, C_2, C_3, etc. are dynamic error coefficients. These are given as

$$C_0 = \int_0^{\infty} f_1(\tau)d\tau = \lim_{s\to 0} F_1(s)$$
$$C_1 = \int_0^{\infty} -\tau\, f_1(\tau)d\tau = \lim_{s\to 0} \frac{dF_1(s)}{ds}$$
$$C_2 = \int_0^{\infty} \frac{\tau^2}{2!}\, f_1(\tau)d\tau = \lim_{s\to 0} \frac{d^2F_1(s)}{ds^2} \quad \text{, and so on…} \tag{8.52}$$
$$C_3 = \int_0^{\infty} -\frac{\tau^3}{3!}\, f_1(\tau)d\tau + = \lim_{s\to 0} \frac{d^3F_1(s)}{ds^3}$$

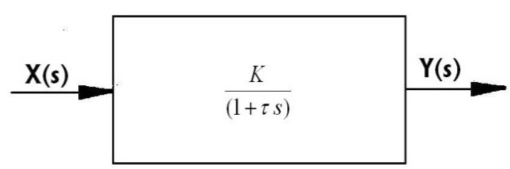
8.8. First-order system:

A Governing differential equation is given by

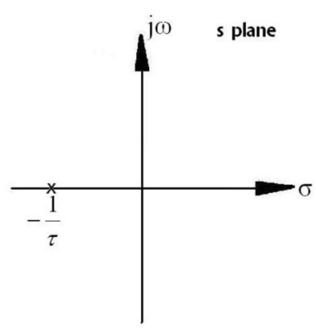
$$y + \tau\,\dot{y} = Kx(t) \tag{8.53}$$

Where, Time constant, sec = τ,
Static sensitivity (units depend on the input and output variables) = K,
y(t) is response of the system and
x(t) is input excitation
The System transfer function is

$$\frac{Y(s)}{X(s)} = G(s) = \frac{K}{(1+\tau\, s)} \tag{8.54}$$



Pole-zero map of a first-order system



Normalized response

In this type of response

- Static components are taken out leaving only the dynamic component
- The dynamic components converge to the same value for different physical systems of the same type or order
- Helps in recognizing typical factors of a system

8.8.1. Impulse input to a first-order system

Governing differential equation

$$y+\tau\dot{y}=Kx_i\delta(t) \tag{8.55}$$

Laplacian of the response

$$Y(s)=\frac{Kx_i}{(1+\tau s)}=\frac{Kx_i}{\tau}\left(\frac{1}{s+\frac{1}{\tau}}\right) \tag{8.56}$$

Time-domain response

$$y(t)=\frac{Kx_i}{\tau}e^{-\frac{t}{\tau}} \tag{8.57}$$

Impulse response function of a first-order system

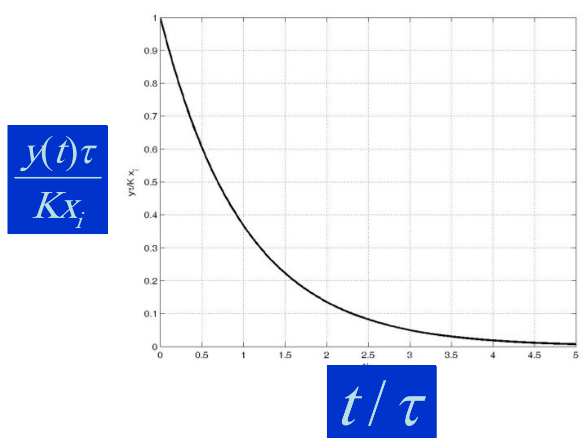
$$h(t)=\frac{K}{\tau}e^{-\frac{t}{\tau}} \tag{8.58}$$

By putting x_i=1 in the response

Response of a first-order system to any force excitation

$$y(t)=\frac{K}{\tau}\int_0^t e^{-\frac{t}{\tau}}F(t-\eta)\,d\eta \tag{8.59}$$

The above equation is called Duhamel's integral. Normalized response of a first-order system to impulse input is shown below.



8.8.2. Step input to a first-order system

Governing differential equation

$$y+\tau\dot{y}=Kx_iu(t) \tag{8.60}$$

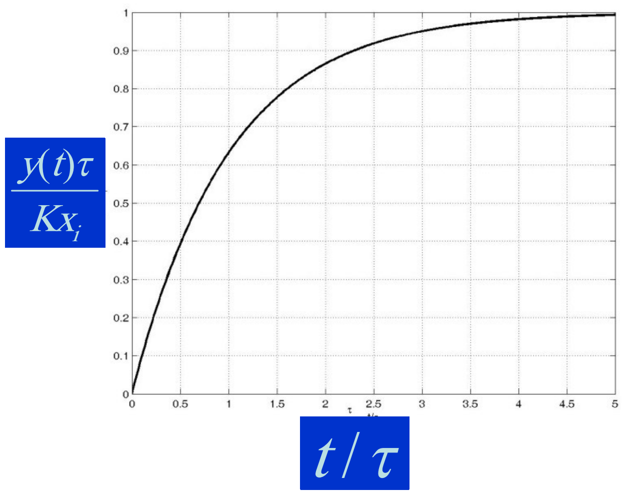
Laplacian of the response

$$Y(s) = \frac{Kx_i}{s(1+\tau s)} = \frac{Kx_i}{s} - \frac{Kx_i}{s + \frac{1}{\tau}} \tag{8.61}$$

Time-domain response

$$y(t) = Kx_i\left(1 - e^{-\frac{t}{\tau}}\right) \tag{8.62}$$

Normalized response of a first-order system to impulse input is shown below.



8.8.3. Ramp input to a first-order system

Governing differential equation

$$y + \tau \dot{y} = Kt \tag{8.63}$$

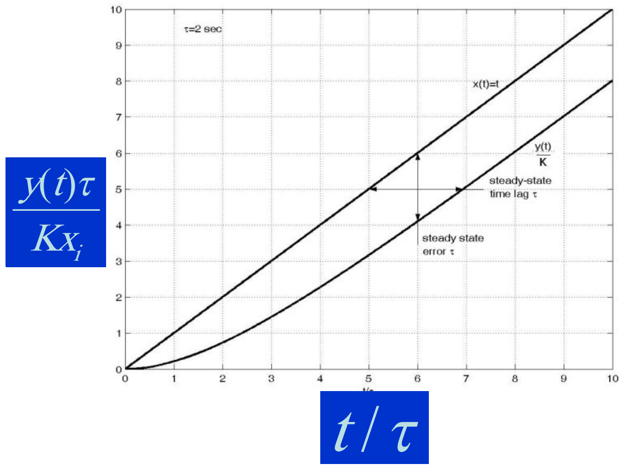
Laplacian of the response

$$Y(s) = \frac{K}{s^2(1+\tau s)} = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau}{s + \frac{1}{\tau}} \tag{8.64}$$

Time-domain response

$$\frac{y(t)}{K} = t - \tau + \tau e^{-\frac{t}{\tau}} \tag{8.65}$$

Normalized response of a first-order system to impulse input is shown below.



8.8.4. Sinusoidal input to a first-order system

Governing differential equation

$$y+\tau\,\dot{y}=KA\sin\omega t \tag{8.66}$$

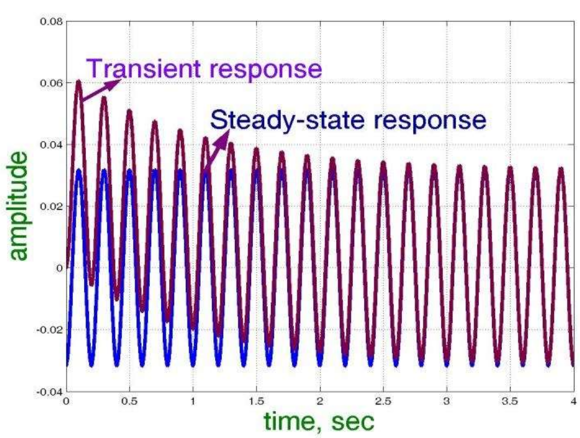
Laplacian of the response

$$Y(s)=\frac{K}{(1+\tau s)}\left(\frac{A\omega}{s^2+\omega^2}\right)=\frac{\omega}{1+(\omega\tau)^2}\left\{\frac{\tau}{s+1/\tau}-\frac{\tau s}{s^2+\omega^2}+\frac{1}{s^2+\omega^2}\right\} \tag{8.67}$$

Time-domain response

$$\frac{y(t)}{KA}=\frac{\omega}{1+(\omega\tau)^2}\left\{\tau e^{-t/\tau}-\tau\cos\omega t+\frac{1}{\omega}\sin\omega t\right\} \tag{8.68}$$

Normalized response of a first-order system to impulse input is shown below.



8.9. Second-order system

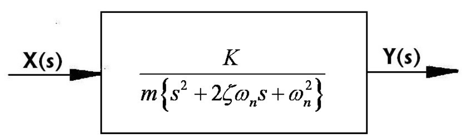
A Governing differential equation is given by

$$m\ddot{y} + c\dot{y} + ky = Kx(t) \tag{8.69}$$

Where, τ = Time constant, sec,
K = Static sensitivity (units depend on the input and output variables),
m = Mass (kg),
c = Damping coefficient (N-s/m),
k = Stiffness (N/m),
y(t) is response of the system and
x(t) is input excitation

The System transfer function is

$$\frac{Y(s)}{X(s)} = \frac{K}{m\left\{s^2 + 2\zeta\omega_n s + \omega_n^2\right\}} \tag{8.70}$$

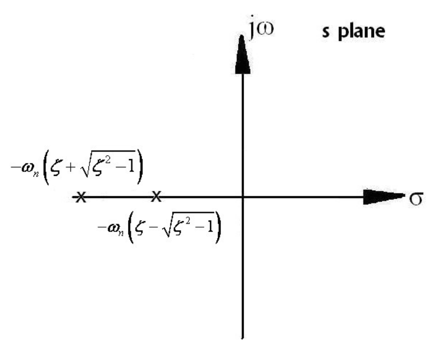


Pole-zero map

(a) ζ>1 over damped
Poles are:

$$s_{1,2} = -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right) \tag{8.71}$$

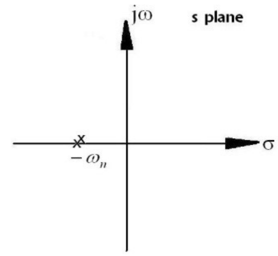
Graphically, the poles of an over damped system is shown as follows.



(b) ζ =1 critically damped
Poles are:

$$s_{1,2} = -\omega_n \tag{8.72}$$

Graphically, the poles of an critically damped system is shown as follows.



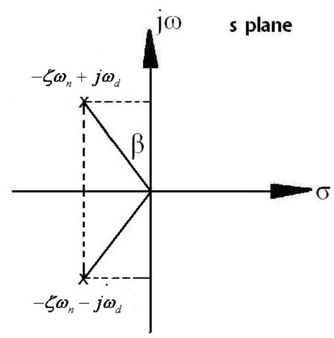
(c) ζ<1 under damped
Poles are:

$$s_{1,2} = -\omega_n\left(\zeta \pm \mathrm{j}\sqrt{1-\zeta^2}\right)$$
$$\Rightarrow s_{1,2} = -\zeta\omega_n \pm j\omega_d \tag{8.73}$$

Where, $\omega_d =$ Damped natural frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{8.74}$$

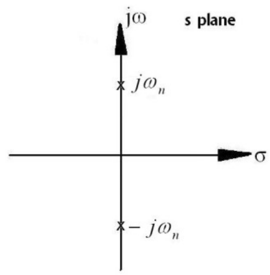
Graphically, the poles of an critically damped system is shown as follows.



Here, $\tan\beta = \dfrac{\zeta}{\sqrt{1-\zeta^2}}$

(d) ζ = 0 un-damped
Poles are:

$$s_{1,2} = -\pm \mathrm{j}\omega_n \tag{8.75}$$



Solved problems:

1. A single degree of freedom spring-mass-damper system has the following data: spring stiffness 20 kN/m; mass 0.05 kg; damping coefficient 20 N-s/m. Determine
   (a) undamped natural frequency in rad/s and Hz
   (b) damping factor
   (c) damped natural frequency n rad/s and Hz.

If the above system is given an initial displacement of 0.1 m, trace the phasor of the system for three cycles of free vibration.

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20\times10^3}{0.05}} = 632.46 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{632.46}{2\pi} = 100.66\,\text{Hz}$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{20\times10^3\times0.05}} = 0.32$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 632.46\sqrt{1-0.32^2} = 600\,\text{rad/s}$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{600}{2\pi} = 95.37\,\text{Hz}$$

$$y(t) = Ae^{-\zeta\omega_n t} = 0.1e^{-0.32\times632.46\,t}$$

2. A second-order system has a damping factor of 0.3 (underdamped system) and an un-damped natural frequency of 10 rad/s. Keeping the damping factor the same, if the un-damped natural frequency is changed to 20 rad/s, locate the new poles of the system? What can you say about the response of the new system?

Solution:

Given, $\omega_{n1} = 10$ rad/s and $\omega_{n2} = 20$ rad/s

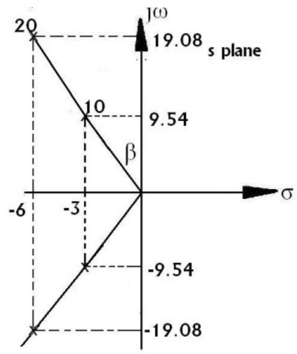
$$\omega_{d_1} = \omega_{n_1}\sqrt{1-\zeta^2} = 10\sqrt{1-0.3^2} = 9.54\,\text{rad/s}$$

$$\omega_{d_2} = \omega_{n_2}\sqrt{1-\zeta^2} = 20\sqrt{1-0.3^2} = 19.08\,\text{rad/s}$$

$$p_{1,2} = -\zeta\omega_{n_1} \pm j\omega_{d_1} = -3 \pm j9.54$$

$p_{3,4} = -\zeta\omega_{n_2} \pm j\omega_{d_2} = -6 \pm j19.08$

$\tan\beta = \dfrac{\zeta}{\sqrt{1-\zeta^2}} = \dfrac{0.3}{\sqrt{1-0.3^2}} = 17.45^o$



8.9.1. Second-order Time Response Specifications with Impulse input

(a) Over damped case (ζ>1)
General equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{Kx_i}{m}\delta(t) \tag{8.76}$$

Laplacian of the output

$$\begin{aligned} Y(s) &= \frac{Kx_i}{m}\left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \\ &= \frac{Kx_i}{2m\omega_n\sqrt{\zeta^2-1}}\left\{\frac{1}{(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})} - \frac{1}{(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}\right\} \end{aligned} \tag{8.77}$$

Time-domain response

$$y(t) = \left[\frac{Kx_i}{m\omega_n\sqrt{\zeta^2-1}}\right]e^{-\zeta\omega_n t}\sinh\left(\omega_n\sqrt{\zeta^2-1}\right)t \tag{8.78}$$

(b) Critically damped case (ζ=1)
General equation

$$\ddot{y} + \omega_n^2 y = \frac{Kx_i}{m}\delta(t) \tag{8.79}$$

Laplacian of the output

$$Y(s) = \frac{Kx_i}{m}\left(\frac{1}{s^2 + \omega_n^2}\right) \tag{8.80}$$

Time-domain response

$$y(t) = \left\{\frac{Kx_i}{m\omega_n}\right\}\omega_n t e^{-\omega_n t} \tag{8.81}$$

(c) Under damped case (ζ<1)

Poles are: $s_{1,2} = -\zeta\omega_n \pm j\omega_d$

General equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{Kx_i}{m}\delta(t) \tag{8.82}$$

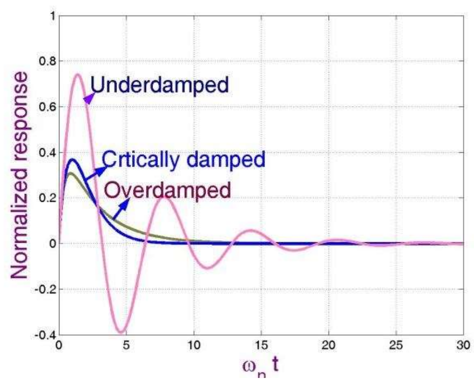
Laplacian of the output

$$Y(s) = \frac{Kx_i}{m}\left\{\frac{1}{(s+\zeta\omega_n + j\omega_d)(s+\zeta\omega_n - j\omega_d)}\right\} \tag{8.83}$$

Time-domain response

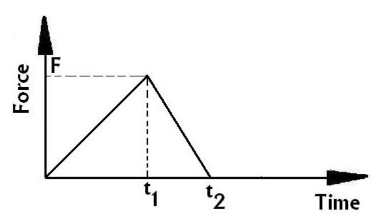
$$y(t) = \left\{\frac{Kx_i}{m\omega_d}\right\} e^{-\zeta\omega_n t}\sin\omega_d t \tag{8.84}$$

Normalized impulse-response of a second-order system with different damping factors are shown graphically as follows.



Solved problems:

3. A second-order system has an un-damped natural frequency of 100 rad/s and a damping factor of 0.3. The value of the coefficient of the second time derivative (that is m) is 5. If the static sensitivity is 10, write down the response (do not solve) for a force excitation shown in the figure in terms of the Duhamel's integral for the following periods of time: 0<t<t1, t1<t<t2 and t>t2.



Solution:

Given, Undamped natural frequency ω_n=100 rad/s

Damping factor ξ =0.3

Coefficient of the second time derivative m=5

Static sensitivity K=10

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 100\sqrt{1-0.3^2} = 95.39 \text{ rad/s}$$

Here,

$$F(t) = F\frac{t}{t_1} \quad ;0 \le t < t_1$$

$$F(t) = \frac{F}{t_2 - t_1}\left(t_2 - t\right) \quad ;t_1 \le t < t_2$$

$$y(t) = \frac{K}{m\omega_d}\int_0^t F(t-\eta)e^{-\zeta\omega_n\eta}\sin\left(\omega_d\eta\right)d\eta$$

$$\Rightarrow y(t) = \frac{10F}{5\times 95.39t_1}\int_0^t e^{-0.3\times 100\eta}\sin\left(95.39\eta\right)(t-\eta)d\eta$$
$$= \frac{0.057F}{t_1}\int_0^t e^{-30\eta}\sin\left(95.39\eta\right)(t-\eta)d\eta \qquad ;0 < t < t_1 ,$$

$$\Rightarrow y(t) = \frac{0.057F}{t_1}\int_0^{t_1} e^{-30\eta}\sin\left(95.39\eta\right)(t-\eta)d\eta$$
$$+\frac{0.057F}{t_2 - t_1}\int_{t_1}^{t} e^{-30\eta}\sin\left(95.39\eta\right)(t_2 - t-\eta)d\eta \qquad ;t_1 < t < t_2 \text{ and}$$

$$\Rightarrow y(t) = \frac{0.057F}{t_1}\int_0^{t_1} e^{-30\eta}\sin\left(95.39\eta\right)(t-\eta)d\eta$$
$$+\frac{0.057F}{t_2 - t_1}\int_{t_1}^{t_2} e^{-30\eta}\sin\left(95.39\eta\right)(t_2 - t-\eta)d\eta \qquad ;t > t_2$$

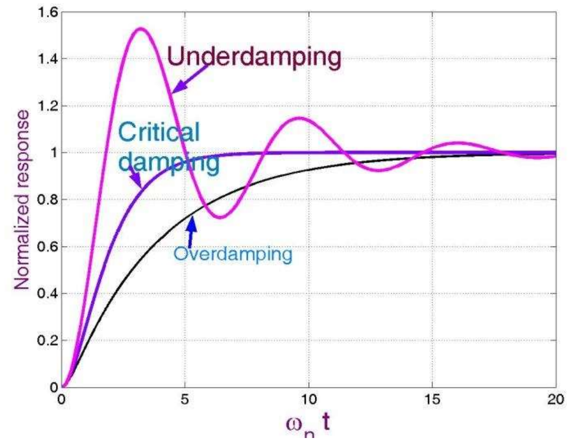
8.9.2. Second-order Time Response Specifications with step input

$$Y(s) = \frac{Kx_i}{m}\left\{\frac{1}{s(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})}\right\} \quad (8.85)$$

$$y(t) = \frac{Kx_i}{m\omega_n^2}\left\{1-e^{-\zeta\omega_n t}\left[\cosh\left(\omega_n\sqrt{\zeta^2-1}\right)t+\frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\left(\omega_n\sqrt{\zeta^2-1}\right)t\right]\right\} \quad (8.86)$$

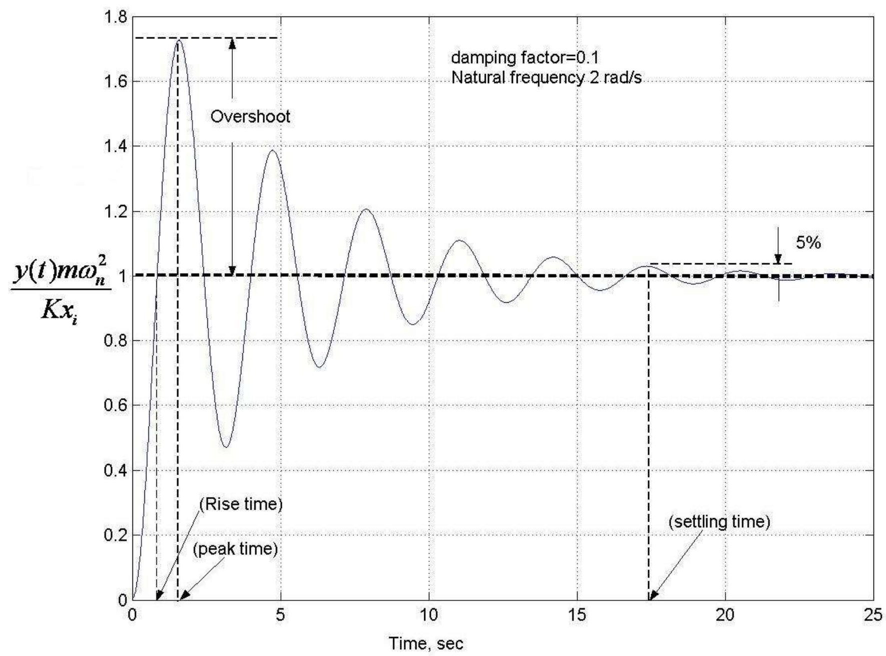
$$Y(s) = \frac{Kx_i}{m}\left\{\frac{1}{s(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}\right\} \quad (8.87)$$

$$y(t) = \frac{Kx_i}{m\omega_n^2}\left\{1-e^{-\zeta\omega_n t}\left[\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]\right\} \quad (8.88)$$



8.10. Time Response Specifications with step-input for under-damped case

For under-damped case, the step-response of a second-order is shown as follows



$$y(t) = \frac{Kx_i}{m\omega_n^2}\left\{1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \varphi)\right\} \tag{8.89}$$

$$\varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \tag{8.90}$$

For this case, different time-domain specifications are described below.

(i) Delay time, t_d

(ii) Rise time, t_r
(iii) Peak time, t_p
(iv) Peak overshoot, M_p
(v) Settling time

For unity step input,
(i)Delay time, t_d: It is the time required to reach 50% of output.

$$y(t_d) = \frac{1}{2} = 1 - \frac{e^{-\zeta\omega_n t_d}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_d + \varphi)$$

$$\Rightarrow t_d = \frac{1+0.7\zeta}{w_n} \tag{8.91}$$

(ii) Rise time, t_r: The time required by the system response to reach from 10% to 90% of the final value for over-damped case, from 0% to 100% of the final value for under-damped case and from 5% to 95% of the critically value for over-damped case.
.

$$y(t_r) = 1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r + \varphi)$$

$$\Rightarrow \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r + \varphi) = 0$$

$$\Rightarrow \omega_d t_r + \varphi = \pi$$

$$\Rightarrow t_r = \frac{\pi - \varphi}{w_d} \tag{8.92}$$

(iii) Peak time, t_p: The time required by the system response to reach the first maximum value.

$$\frac{dy(t_p)}{dt} = 0$$

$$\Rightarrow \frac{d\left[1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p + \varphi)\right]}{dt} = 0$$

$$\Rightarrow \frac{d\left[-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p + \varphi)\right]}{dt} = 0$$

$$\Rightarrow w_d t_p + \varphi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = n\pi + \varphi \text{ ; where } n = 1,2,3,...$$

For n=1,

$$\Rightarrow w_d t_p = n\pi$$

$$\Rightarrow t_p = \frac{n\pi}{w_d} \tag{8.93}$$

(iv) Peak overshoot, M_p: It is the time required to reach 50% of output.

$$M_p(\%) = 100 \times \frac{y(t_p) - 1}{1}$$

$$\Rightarrow M_p(\%) = 100 \times \left[1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \varphi) - 1\right]$$

$$\Rightarrow M_p(\%) = 100 \times \left[-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \varphi)\right] = 100 \times \left[-\frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \varphi)\right]$$

$$\Rightarrow M_p(\%) = 100 \times \left[-\frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\omega_d \frac{\pi\zeta}{\sqrt{1-\zeta^2}} + \varphi)\right] = 100 \times \left[-\frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\pi + \varphi)\right]$$

$$\Rightarrow M_p(\%) = 100 \times \left[\frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\varphi\right] = 100 \times \left[\frac{e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2}\right]$$

$$\Rightarrow M_p(\%) = 100 \times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{8.94}$$

(iv) Settling time, t_s: It is the time taken by the system response to settle down and stay with in $\pm 2\%$ or $\pm 5\%$ its final value.
For $\pm 2\%$ error band,

$$t_s = \frac{4}{\zeta\, w_n} \tag{8.95}$$

For $\pm 5\%$ error band,

$$t_s = \frac{3}{\zeta\, w_n} \tag{8.96}$$

Sl. No.	**Time Specifications**	
	Type	**Formula**
1	Delay time	$t_d = \frac{1 + 0.7\zeta}{w_n}$
2	Rise time	$t_r = \frac{\pi - \varphi}{w_d}$
3	Peak time	$t_p = \frac{\pi}{w_d}$
4	Maximum overshoot	$M_p(\%) = 100 \times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$
5	Settling time	$t_s = \frac{4}{\zeta\, w_n}$

Solved Problems:

1. Consider the system shown in Figure 1. To improve the performance of the system a feedback is added to this system, which results in Figure 2. Determine the value of K so that the damping ratio of the new system is 0.4. Compare the overshoot, rise time, peak time and settling time and the nominal value of the systems shown in Figures 1 and 2.

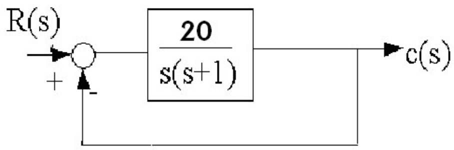


Figure 1

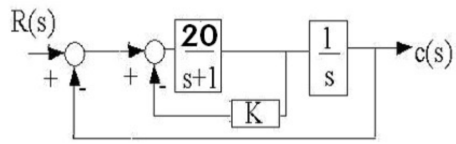


Figure 2

Solution:

For Figure 1,

$$\frac{c(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\dfrac{20}{s(s+1)}}{1+\dfrac{20}{s(s+1)}}=\frac{20}{s^2+s+20}$$

Here, $\omega_n^2=20$ and $2\zeta\omega_n=1$

$$\omega_n=\sqrt{20}\text{ rad/s and }\zeta=\frac{1}{2\omega_n}=\frac{1}{2\times\sqrt{20}}=0.112$$

For Figure 2,

$$\frac{c(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\dfrac{20}{s(s+1+20K)}}{1+\dfrac{20}{s(s+1+20K)}}=\frac{20}{s^2+(1+20K)s+20}$$

Here, $\omega_n^2=20$ and $2\zeta\omega_n=1+20\text{K}$

$$\omega_n=\sqrt{20}\text{ rad/s}$$

But, given that $\zeta=\dfrac{1+20K}{2\omega_n}=\dfrac{1+20K}{2\sqrt{20}}=0.4$

$\Rightarrow K=0.128$

Transient characteristics of Figures 1 and 2

CharacteristicS	**Figure 1**	**Figure 2**
Overshoot, M_p	70%	25%
Rise time, $t_{r,}$ sec	0.38	0.48
Peak time, t_p, sec	0.71	0.77
Settling time (2%), sec	8	2.24
Steady-state value, c_∞	1.0	1.0

Equation Chapter (Next) Section 1

1.1. Transient Response using MATLAB

Program 1: Find the step response for the following system $\frac{C(s)}{R(s)} = \frac{3s+20}{s^2+5s+36}$

Solution:

```
>> num=[3 20]
num=
3        20
>> den=[1 5 36]
den=
1        5        36
>>sys=tf(num,den)
Transfer function:
      3s+20
--------------------
   s^2+5s+36
>>step(sys)
```

Program 2: Find the step response for the following system $\frac{C(s)}{R(s)} = \frac{20}{s^2+4s+25}$

Solution:

```
>> num=[20]
num=
         20
>> den=[1 425]
den=
         1        4        25
>>sys=tf(num,den)
Transfer function:
20
--------------------
   s^2+4s+25
>>step(sys)
```

2. Stability

2.1. Concept of stability

Stability is a very important characteristic of the transient performance of a system. Any working system is designed considering its stability. Therefore, all instruments are stable with in a boundary of parameter variations.

A linear time invariant (LTI) system is stable if the following two conditions are satisfied.

(i) **Notion-1:** When the system is excited by a bounded input, output is also bounded.

Proof:

A SISO system is given by

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_n} \tag{9.1}$$

So,

$$c(t) = \alpha^{-1}\left[G(s)R(s)\right] \tag{9.2}$$

Using convolution integral method

$$c(t) = \int_0^\infty g(\tau)r(t-\tau)d\tau \tag{9.3}$$

$g(\tau) = \alpha^{-1}G(s) =$ impulse response of the system

Taking absolute value in both sides,

$$|c(t)| = \left|\int_0^\infty g(\tau)r(t-\tau)d\tau\right| \tag{9.4}$$

Since, the absolute value of integral is not greater than the integral of absolute value of the integrand

$$\begin{aligned} |c(t)| &\le \int_0^\infty |g(\tau)r(t-\tau)\,d\tau| \\ \Rightarrow |c(t)| &\le \int_0^\infty |g(\tau)r(t-\tau)|d\tau \\ \Rightarrow |c(t)| &\le \int_0^\infty |g(\tau)||r(t-\tau)|d\tau \end{aligned} \tag{9.5}$$

Let, r(t) and c(t) are bounded as follows.

$$\begin{aligned} |r(t)| &\le M_1 < \infty \\ |c(t)| &\le M_2 < \infty \end{aligned} \tag{9.6}$$

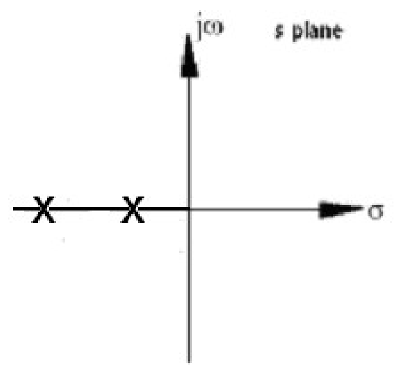
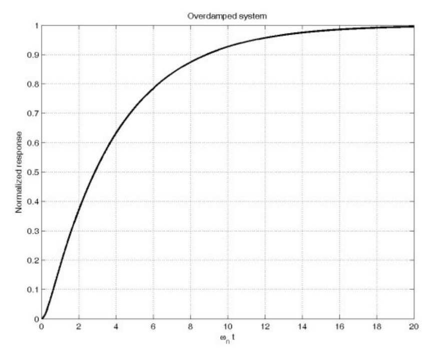
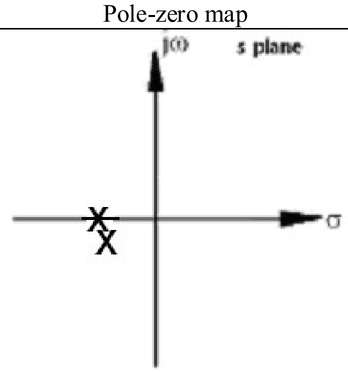
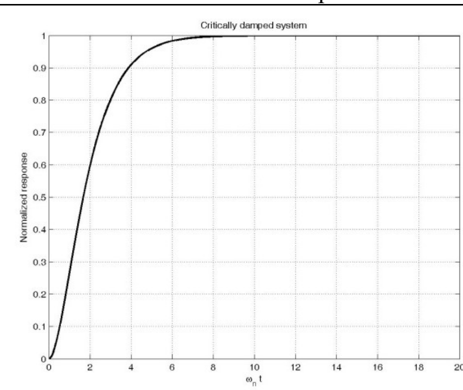
Then,

$$|c(t)| \le M_1 \int_0^{\infty} |g(\tau)| d\tau \le M_2 \tag{9.7}$$

Hence, first notion of stability is satisfied if $\int_0^{\infty} |g(\tau)| d\tau$ is finite or integrable.

(ii) **Notion-2:** In the absence of the input, the output tends towards zero irrespective of initial conditions. This type of stability is called asymptotic stability.

2.2. Effect of location of poles on stability

Pole-zero map	**Normalized response**
Over-damped close-loop poles	
	
Critically damped close-loop poles	
Pole-zero map	Normalized response
	
Under-dampedclose-loop poles	
Pole-zero map	Normalized response

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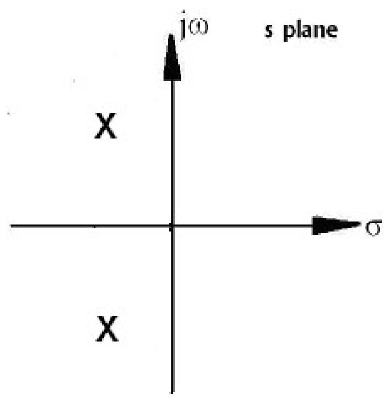
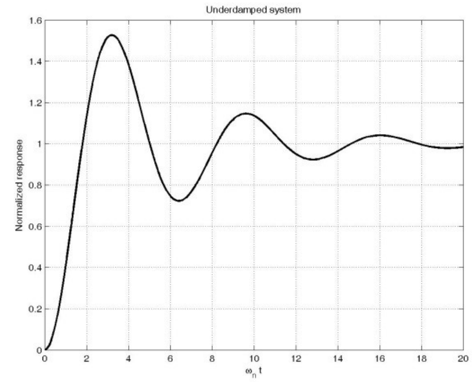
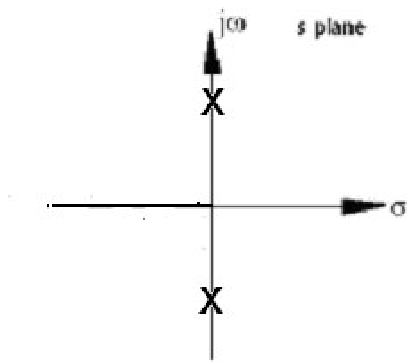
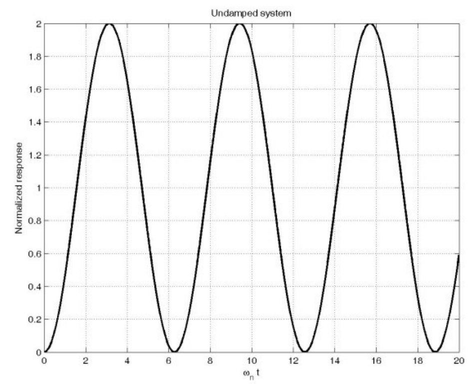
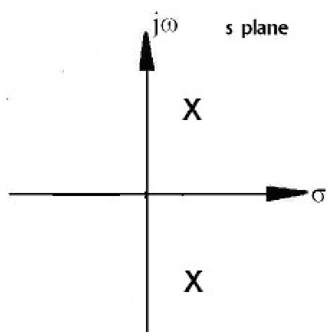
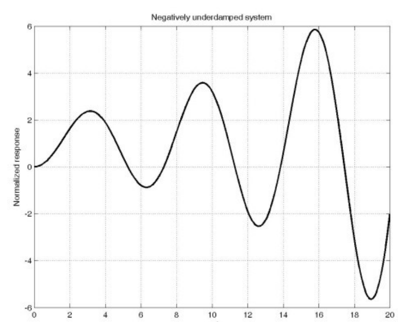
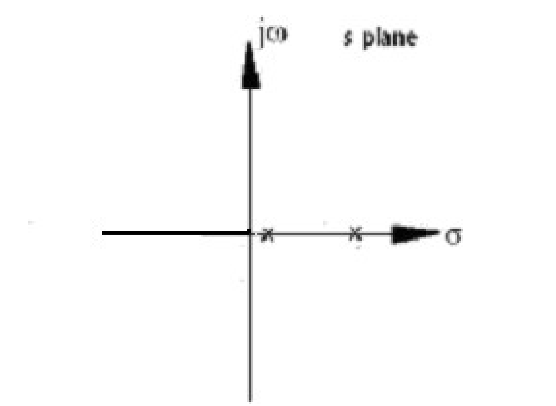
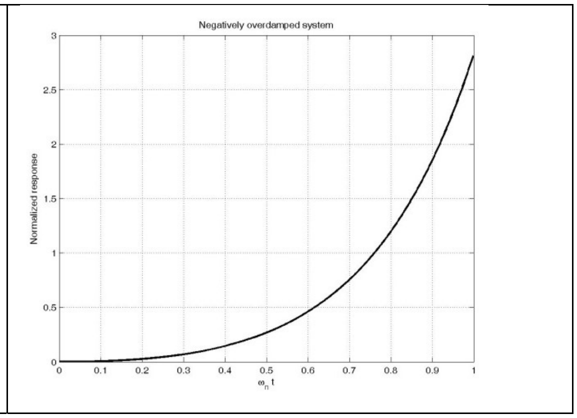
<tr><td>
</td><td>
</td></tr>
<tr><td colspan="2">Un-dampedclose-loop poles</td></tr>
<tr><td>Pole-zero map</td><td>Normalized response</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td colspan="2">Negative Under-dampedclose-loop poles</td></tr>
<tr><td>Pole-zero map</td><td>Normalized response</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td colspan="2">Negative Over-dampedclose-loop poles</td></tr>
<tr><td>Pole-zero map</td><td>Normalized response</td></tr>
</table>





2.3. Closed-loop poles on the imaginary axis

Closed-loop can be located by replace the denominator of the close-loop response with s=jω.

Example:

1. Determine the close-loop poles on the imaginary axis of a system given below.

$$G(s) = \frac{K}{s(s+1)}$$

Solution:

Characteristics equation, $B(s) = s^2 + s + K = 0$

Replacing $s = jw$

$B(j\omega) = (j\omega)^2 + (j\omega) + K = 0$

$\Rightarrow (K - \omega^2) + j\omega = 0$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\omega = \sqrt{K}$ and $\omega = 0$

Therefore, Closed-loop poles do not cross the imaginary axis.

2. Determinetheclose the imaginary axis of a system given below.

$B(s) = s^3 + 6s^2 + 8s + K = 0$.

Solution:

Characteristics equation,

$B(j\omega) = (j\omega)^3 + 6(j\omega)^2 + 8j\omega + K = 0$

$\Rightarrow (K - 6\omega^2) + j(8\omega - \omega^3) = 0$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\omega = \pm\sqrt{8}$ rad/s and $K = 6\omega^2 = 48$

Therefore, Close-loop poles cross the imaginary axis for K>48.