1.7 Limits and their properties, definition and theory

-Properties of limits

1. $\lim_{X \to a} x = a$

2. $\lim_{x \to a} c = c$

3. $\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a}\big(f(x)g(x)\big) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

5. $\lim_{x \to a}\big(f(x) \pm g(x)\big) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

6. $\lim_{x \to a}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

7. $\lim_{x \to 0} \dfrac{1}{x} = \infty$

8. $\lim_{x \to \infty} \dfrac{1}{x} = 0$

9. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

10. $\lim_{x \to 0} \dfrac{x}{\sin x} = 1$

11. $\lim_{x \to \infty} \dfrac{\sin x}{x} = 0$

Example 1: Find $\lim_{x \to 0} x^2 + 3x + 5$

Solution//

=0+3*0+5=5

Example 2: Find $\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}$

Solution //

$$\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}=\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$$

$$=\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}+1)(\sqrt{x}-1)}=\lim_{x\to 1}\frac{(2x-3)}{(2x+3)(\sqrt{x}+1)}$$

$$\lim_{x\to 1}\frac{(2*1-3)}{(2*1+3)(\sqrt{1}+1)}=-1/10$$

Example 3: Find $\lim\limits_{x\to 2}\dfrac{(x^2-4)}{(x^3-8)}$

Solution //

$$\lim_{x\to 2}\frac{(x^2-4)}{(x^3-8)}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)}$$

$$=\lim_{x\to 2}\frac{(x+2)}{(x^2+2x+4)}=\lim_{x\to 2}\frac{(2+2)}{(2^2+2*2+4)}$$

$$=1/3$$

Example 4: Find $\lim\limits_{x\to 2}\dfrac{(4-x^2)}{(3-\sqrt{x^2+5})}$

Solution /

$$\lim_{x\to 2}\frac{(4-x^2)}{(3-\sqrt{x^2+5})}*\frac{(3+\sqrt{x^2+5})}{(3+\sqrt{x^2+5})}=\lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})*(3+\sqrt{x^2+5})}$$

$$= \lim_{x \to 2} \frac{(4 - x^2)(3 + \sqrt{x^2 + 5})}{9 - (x^2 + 5)} =$$

$$= \lim_{x \to 2} \frac{(4 - x^2)(3 + \sqrt{x^2 + 5})}{(4 - x^2)} = \left(3 + \sqrt{x^2 + 5}\right)$$

$$= 3 + \sqrt{4 + 5} = 6$$

Example 5: Find $\lim_{x \to \infty} \frac{(x+2)}{\sqrt{x^2+1}}$

Solution //

$$\lim_{x \to \infty} \frac{(\frac{x}{x^2}+\frac{2}{x^2})}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}} = \lim_{x \to \infty} \frac{(\frac{1}{x}+\frac{2}{x^2})}{\sqrt{1+\frac{1}{x^2}}} = \frac{0}{1} = 0$$

Example 6: Find $\lim_{x \to 2}(\frac{1}{x-2} - \frac{4}{x^2-4})$

Solution //

$$\lim_{x \to 2}(\frac{1}{x - 2} - \frac{4}{x^2 - 4}) = \lim_{x \to 2}(\frac{1}{x - 2} - \frac{4}{(x - 2)(x + 2)})$$

$$\lim_{x \to 2}(\frac{1}{x - 2}\left(1 - \frac{4}{(x + 2)}\right) = \lim_{x \to 2}(\frac{(x + 2 - 4)}{(x - 2)(x + 2)}$$

$$= \lim_{x \to 2}(\frac{(x-2)}{(x-2)(x+2)})$$

$$= \frac{1}{2 + 2} = 1/4$$

Example 7: Find $\lim\limits_{h \to 0}(\frac{\sin(a+h)-\sin a}{h})$

Solution //

$$\lim_{h \to 0}(\frac{\sin(a)\cos(h) + \cos(a)\sin(h) - \sin(a)}{h})$$

$$\lim_{h \to 0}(\frac{\sin(a)((\cos(h) - 1) + \cos(a)\sin(h)}{h})$$

$$\lim_{h \to 0}(\frac{-\sin(a)(1 - \cos(h) + \cos(a)\sin(h)}{h}$$

$$\lim_{h \to 0}(\frac{-\sin(a)(1 - \cos(h)}{h} + \lim_{h \to 0}(\frac{\sin(h)\cos(a)}{h} = \cos a$$

Example 8: Find $\lim\limits_{x \to 0}\frac{(1-\sqrt{\cos x})}{x^2}$

Solution //

$$\lim_{x \to 0}\frac{(1 - \sqrt{\cos x})}{x^2} * \frac{(1 + \sqrt{\cos x})}{(1 + \sqrt{\cos x})}$$

$$= \lim_{x \to 0}\frac{1 - \cos x}{x^2(1 + \sqrt{\cos x})} = \lim_{x \to 0}\frac{1 - \cos x}{x^2(1 + \sqrt{\cos x})} * \frac{1 + \cos x}{(1 + \cos x)}$$

$$= \lim_{x \to 0}\frac{1 - \cos^2 x}{x^2((1 + \cos x)(1 + \sqrt{\cos x})}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x^2\left((1+\cos x)(1+\sqrt{\cos x})\right)}$$

$$=1/4$$

Example 9: Find $\lim\limits_{x\to\pi}(\frac{\sin(x)}{x-\pi})$

Solution /

Y=x-π → $x = y + \pi . y = 0$

$$\lim_{y\to 0}(\frac{\sin(y+\pi)}{y}) = \lim_{y\to 0}(\frac{\sin(y)\cos(\pi) + \sin(\pi)\cos(y)}{y})$$

$$= -1$$

Exercises 1: Find $\lim\limits_{x\to 0}\sin(\frac{\pi}{2}\cos(\tan X\ )$

2: Find $\lim\limits_{x\to 1}\frac{(\sqrt{x}-2)}{(x-2)}$

3: Find $\lim\limits_{x\to 1}\frac{(x^2-9)}{(\sqrt{X+3})}$

4: Find $\lim\limits_{x\to\infty}(\sqrt{x^2+1}) - X)$

5: Find $\lim\limits_{x\to 0}\frac{(X*\sin X)}{(2-2\cos X)}$

6: Find $\lim\limits_{x\to\infty}\frac{(x^2+2X+1)^{50}}{(x^2+1)^{100}}$