Al-Mustaqbal University
Department of Biomedical Engineering
Third Stage / 1st Course
"Transport Phenomena for BME"
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Lecture 5

"Continuity Equation - Bernoulli Equation"

Conservation of Mass (Continuity Equation)

In fluid dynamics, fluids are moved from one place to other by means of mechanical devices such as pumps or blowers, by gravity head, or by pressure which flow through systems of piping and/or process equipment.

To begin, we consider a conceptual 3-D space or Control Volume the law of Conservation of Mass (also known as the Continuity Equation) states that any change in mass within this control volume must be equal to the mass of fluid that enters the volume (Mass In) minus the mass that exits (Mass Out).

Therefore, for a given period of time

(Rate of Mass In) – (Rate of Mass Out) = Rate of Change of Mass within the Tube

For liquids in general and blood in particular, a very good assumption is that the fluid is incompressible; that is, its density (ρ) is constant. This, together with the fixed size of the Control Volume, causes the time rate of change of mass within the control volume to be zero.

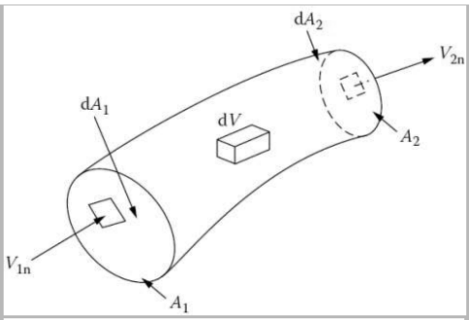


Figure 1. Mass flux balance

$$\dot{m}_{in} - \dot{m}_{out} = {dm}/{dt} = 0$$

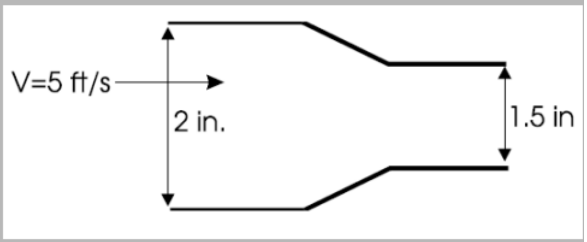
Also, if the flow is steady and there are only two surfaces across which fluid flows (Surfaces 1 and 2), then the mean velocities $\bar{V}$ and cross-sectional areas A can be related as

$$A_1\bar{V}_1\rho_1 = A_2\bar{V}_2\rho_2$$

$$\rho_1 = \rho_2 \text{ Incompressible fluid}$$

$$A_1\bar{V}_1 = A_2\bar{V}_2 = Q\ (constant)$$

Example 1: Steady state flow of water through a nozzle. The diameter at the inlet is 2 in., and the diameter at the exit is 1.5 in. The average inlet velocity is 5 ft/s. What is the average velocity at the exit?



Because the flow is steady, the continuity equation is

$$A_1\bar{V}_1 = A_2\bar{V}_2$$

$$A_1\bar{V}_1 = \frac{\pi}{4} * \left(\frac{2}{12}\right)^2 * 5 = 0.109$$

$$A_2\bar{V}_2 = \frac{\pi}{4} * \left(\frac{1.5}{12}\right)^2 * \bar{V}_2$$

$0.109 = 0.01228\ \bar{V}_2$

$\bar{V}_2 = 8.89$ ft/s

Average velocity

The fluid velocity in a pipe changes from zero at the wall because of the no-slip condition to a maximum at the pipe center. In fluid flow, it is convenient to work with an average velocity Vavg, which remains constant in incompressible flow when the cross-sectional area of the pipe is constant

The value of the average velocity Vavg at some streamwise cross-section is determined from the requirement that the conservation of mass principle be satisfied (Figure 2).

That is,

$$\dot{m} = \rho V_{avg} A_c = \int_{A_c} \rho u(r)\, dA_c$$

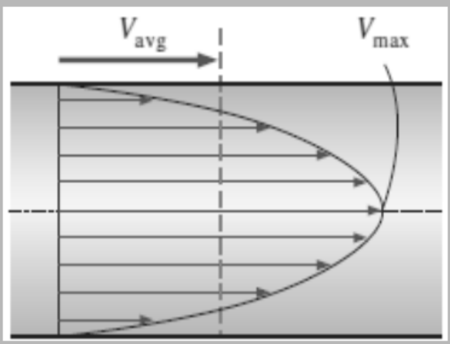


Figure 2. Average velocity V_{avg}

Bernoulli Equation

The total energy of a fluid in motion consists of the following components: -

Potential Energy (PE)

This is the energy that a fluid has because of its position in the earth's field of gravity. The work required to raise a unit mass of fluid to a height (z) above a datum line is (zg), where (g) is gravitational acceleration. This work is equal to the potential energy per unit mass of fluid above the datum line.

Kinetic Energy (KE)

This is the energy associated with the physical state of fluid motion. The kinetic energy of unit mass of the fluid is (u2/2), where (u) is the linear velocity of the fluid relative to some fixed body.

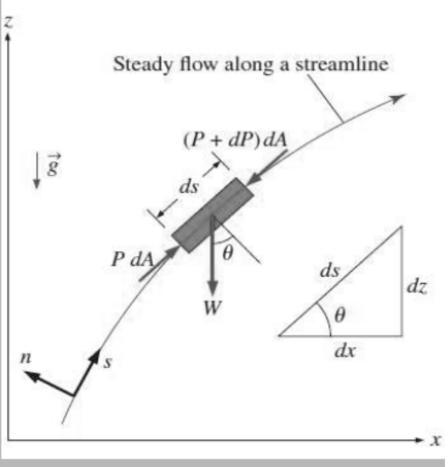
Pressure Energy (flow energy)

This is the energy or work required to introduce the fluid into the system without a change in volume. If (P) is the pressure and (V) is the volume of a mass (m3) of fluid, then (P V /m) is the pressure energy per unit mass of fluid. The ratio (V /m) is the fluid density (1/ρ).

Bernoulli equation

The Bernoulli equation is an approximate relation between pressure, velocity, and elevation, and is valid in regions of steady, incompressible flow where net frictional forces are negligible.

E =zg + P/ ρ + u2/2



Derivation of the Bernoulli Equation

Consider the motion of a fluid particle in a flow field in steady flow. Applying Newton's second law (which is referred to as the linear momentum equation in fluid mechanics) in the s-direction on a particle moving along a streamline gives

$$\sum F_s = ma_s \qquad (1)$$

Steady flow: $$\int \frac{dP}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)}$$

Since the last two terms are exact differentials. In the case of incompressible flow, the first term also becomes an exact differential, and integration gives

Steady, incompressible flow: $$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)}$$

This is the famous Bernoulli equation, which is commonly used in fluid mechanics for steady, incompressible flow along a streamline in inviscid regions of flow. The Bernoulli equation was first stated in words by the Swiss mathematician Daniel Bernoulli (1700–1782) ,The Bernoulli equation can also be written between any two points on the same streamline as

Steady, incompressible flow: $$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

We recognize $V^2/2$ as kinetic energy, gz as potential energy, and P/ρ as flow energy, all per unit mass. Therefore, the Bernoulli equation can be viewed as an expression of mechanical energy balance and can be stated as

The sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline during steady flow when compressibility and frictional effects are negligible.

Limitations on the Use of the Bernoulli Equation

1. Steady flow
2. Negligible viscous effects
3. No shaft work
4. Incompressible flow
5. Negligible heat transfer
6. Flow along a streamline

Modification of Bernoulli's Equation

1. Correction of the kinetic energy term

The velocity in kinetic energy term is the mean linear velocity in the pipe. To account the effect of the velocity distribution across the pipe [(α) dimensionless correction factor] is used. For a circular cross sectional pipe:

α = 2 for laminar flow
α = 1.05 for turbulent flow

$$\frac{P_1}{\rho} + \propto_1 \frac{{V_1}^2}{2} + Z_1 g = \frac{P_2}{\rho} + \propto_2 \frac{{V_2}^2}{2} + Z_2 g$$

1. Modification for real fluid

The real fluids are viscous and hence offer resistance to flow. Friction appears wherever the fluid flow is surrounding by solid boundary. Friction can be defined as the amount of mechanical energy irreversibly converted into heat in a flow in stream. As a result of that the total energy is always decrease in the flow direction i.e. ($E_2 < E_1$). Therefore $E_1 = E_2 + F$, where F is the energy losses due to friction ($J/kg \equiv m^2/s^2$)

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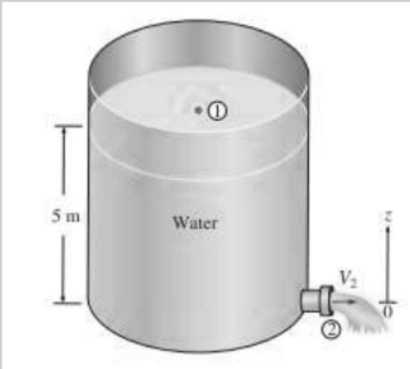
$$\frac{P_1}{\rho} + \propto_1 \frac{{V_1}^2}{2} + Z_1 g = \frac{P_2}{\rho} + \propto_2 \frac{{V_2}^2}{2} + Z_2 g + F$$

Example 1: A large tank open to the atmosphere is filled with water to a height of 5 m from the outlet tap (Fig. 1). A tap near the bottom of the tank is now opened, and water flows out from the smooth and rounded outlet. Determine the maximum water velocity at the outlet.

(Fig. 1).



SOLUTION A tap near the bottom of a tank is opened. The maximum exit velocity of water from the tank is to be determined. Assumptions 1 The flow is incompressible. 2 The water drains slowly enough that the flow can be approximated as steady . 3 Irreversible losses in the tap region are neglected.

$$\frac{P}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P}{\rho g} + \frac{V_2^2}{2g} + z_2 \quad \rightarrow \quad z_1 = \frac{V_2^2}{2g}$$

($V_1^2 \approx 0$, $z_2 = 0$)

Solving for V_2 and substituting,

$$V_2 = \sqrt{2gz_1} = \sqrt{2(9.81\ \text{m/s}^2)(5\ \text{m})} = 9.9\ \text{m/s}$$

Example 2: A piezometer and a Pitot tube are tapped into a horizontal water pipe, as shown in Fig. 2, to measure static and stagnation pressures. For the indicated water column heights, determine the velocity at the center of the pipe

$$P_1 = \rho g(h_1 + h_2)$$

$$P_2 = \rho g(h_1 + h_2 + h_3)$$

Noting that $z_1 = z_2$, and point 2 is a stagnation point and thus $V_2 = 0$, the application of the Bernoulli equation between points 1 and 2 gives

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \quad \rightarrow \quad \frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g}$$

Substituting the P_1 and P_2 expressions gives

$$\frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g} = \frac{\rho g(h_1 + h_2 + h_3) - \rho g(h_1 + h_2)}{\rho g} = h_3$$

Solving for V_1 and substituting,

$$V_1 = \sqrt{2gh_3} = \sqrt{2(9.81\ \text{m/s}^2)(0.12\ \text{m})} = 1.53\ \text{m/s}$$

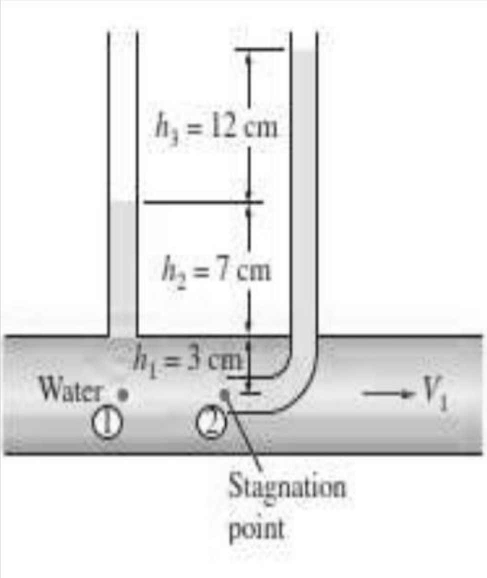


Fig. 2