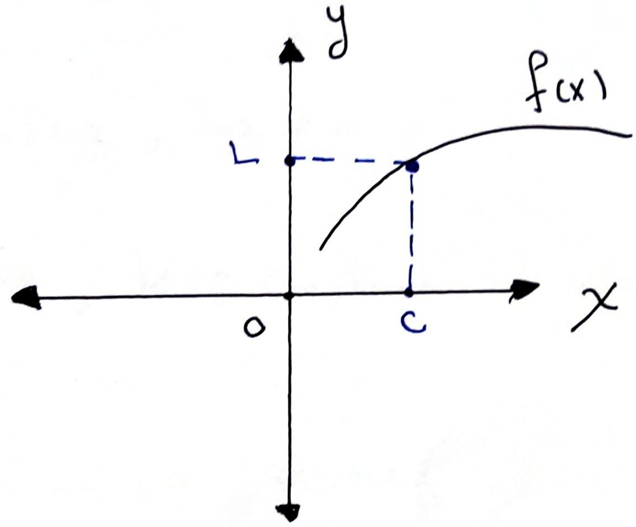
# Lecture One :

## Limits :-

When the values of a function $f(x)$ approach the value $L$ as $x$ approaches $c$, we say that $f(x)$ has limit $L$ as $x$ approaches $c$.

OR: $\lim_{x \to c} f(x) = L$




* Notes

1. For identify function ( $f(x) = x$ )

$$\lim_{x \to c} x = c$$

2. For constant function ( $f(x) = k$ )

$$\lim_{x \to c} k = k$$

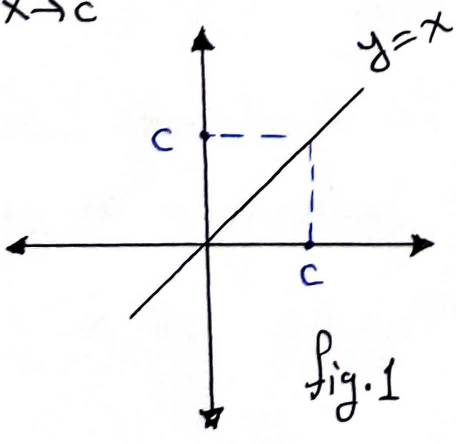



fig.1

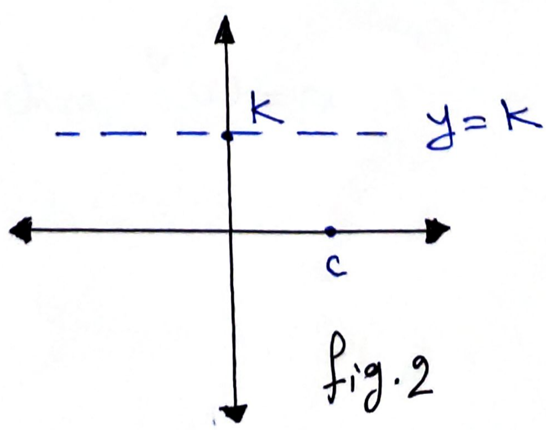



fig.2

# Properties of Limits :-

- For $\lim\limits_{x \to c} f_1(x) = L_1$ & $\lim\limits_{x \to c} f_2(x) = L_2$, then :

  - $\lim\limits_{x \to c} \left[ f_1(x) \mp f_2(x) \right] = L_1 \mp L_2$
  - $\lim\limits_{x \to c} \left[ f_1(x) \cdot f_2(x) \right] = L_1 \cdot L_2$
  - $\lim\limits_{x \to c} \left[ f_1(x) / f_2(x) \right] = L_1 / L_2$, $L_2 \neq 0$
  - $\lim\limits_{x \to c} \left[ k \cdot f_1(x) \right] = k \cdot L_1$, $k$ : constant

- If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is any polynomial function :

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0$$

- If $f(x)$ and $g(x)$ are polynomials :

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad g(c) \neq 0$$

Ex : Find the limit of the function $f(x) = x + 1$ as $x$ approaches 3 ?

Sol: $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to 3} (x+1) = \lim\limits_{x \to 3} x + \lim\limits_{x \to 3} 1$

$= 3 + 1 = 4$

Find the limits :

(1)

$$\lim_{x \to 5} \frac{4}{x-7} = \frac{4}{5-7} = \frac{4}{-2} = -2$$

(2)

$$\lim_{x \to 2} \frac{x^2-7x+10}{x-2}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x-5)}{\cancel{(x-2)}} = \lim_{x \to 2} x-5 = 2-5 = -3$$

(3)

$$f(x) = \begin{cases} 3-x & , x < 2 \\ \frac{x}{2}+1 & , x > 2 \end{cases}$$

a- Find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$.

b- Does $\lim_{x \to 2^-} f(x)$ exist ? why ?

Sol :

a - $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(\frac{x}{2}+1\right) = \frac{2}{2}+1 = 2$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3-x) = 3-2 = 1$

$\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x) \Rightarrow$ limit doesn't exist

## The Sandwich Theorem and $\frac{\sin\theta}{\theta}$ :

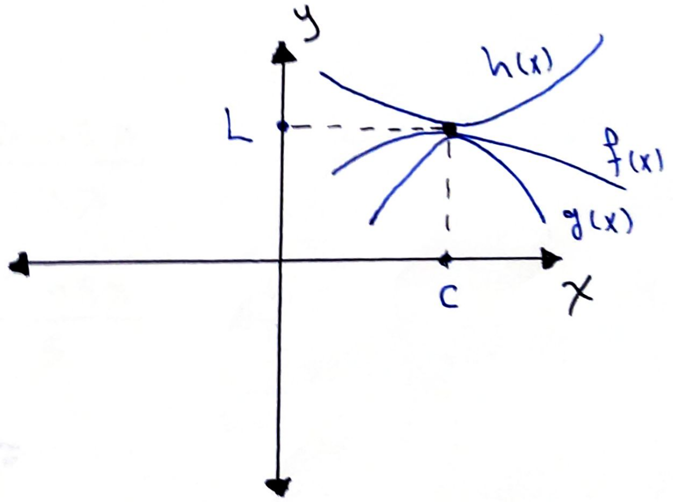
$f(x)$, $h(x)$, $g(x)$

if $g(x) \leq f(x) \leq h(x)$

The Sandwich theorem is :

for all $x \neq c$ & $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

then $\lim_{x \to c} f(x) = L$




### The problem of $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$

If we substitute $\theta = 0$ in $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$, we will get indeterminate form $\frac{0}{0}$, but using Sandwich theorem :

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Find: $\lim_{x \to 0} \frac{\sin 3x}{x}$

Let $3x = \theta \rightarrow x = \frac{\theta}{3}$

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta/3} = \lim_{\theta \to 0} 3 \frac{\sin\theta}{\theta} = 3 * 1 = 3$$

$$\lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{x/x}{\sin x/x} = 1/1 = 1$$

$$\lim_{x\to 0} \frac{\sin x}{2x^2-x} = \lim_{x\to 0} \frac{\sin x}{x(2x-1)} = \lim_{x\to 0} \frac{\sin x}{x} * \frac{1}{2x-1}$$

$$= 1 * \frac{1}{-1} = -1$$

$$\lim_{x\to 0} \frac{x+\sin x}{x} = \lim_{x\to 0} \frac{x}{x} + \frac{\sin x}{x} = 1+1 = 2$$

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{x\to 0} \frac{5}{5} * \frac{\sin 5x}{3x}$$

$$= \lim_{x\to 0} \frac{5}{3} * \frac{\sin 5x}{5}$$

$$= \frac{5}{3} * 1 = \frac{5}{3}$$

$$\lim_{x\to 0} \frac{\tan 2x}{x} = \lim_{x\to 0} \frac{\sin 2x}{x(\cos 2x)} = \lim_{x\to 0} \frac{2\sin x \cos x}{x \cos 2x}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{2\cos x}{\cos 2x} = 1 * \frac{2(1)}{1} = 2$$

H.W

$$\lim_{x\to 0} \frac{\cos 2x - 1}{\cos x - 1}$$

Find :-

$$\lim_{X\to -2} \frac{\frac{1}{x}+\frac{1}{2}}{X^3+8}$$

$$A^3+B^2$$
$$=(A+B)(A^2-AB+B^2)$$

$$= \lim_{X\to -2} \frac{((X+2)/2X)}{(X+2)(X^2-2X+4)}$$

$$= \lim_{X\to -2} \frac{1}{2X(X^2-2X+4)}$$

$$= \frac{1}{2(-2)(-2^2(-2*-2)+4)} = -1/48$$

$$\lim_{X\to 4} \frac{3-\sqrt{X+5}}{X-4}$$

$$\lim_{X\to 4} \frac{3-\sqrt{X+5}}{X-4} * \frac{3+\sqrt{X+5}}{3+\sqrt{X+5}}$$

$$\lim_{X\to 4} \frac{9-(X+5)}{(X-4)(3+\sqrt{X+5})}$$

$$\lim_{X\to 4} \frac{4-x}{(X-4)(3+\sqrt{X+5})} = \lim_{X\to 4} \frac{-(X-4)}{(X-4)(3+\sqrt{X+5})}$$

$$= \lim_{X\to 4} \frac{-1}{3+\sqrt{X+5}} = \frac{-1}{3+\sqrt{4+5}} = -1/6$$

$$\lim_{x\to 1} \frac{x^3-1}{(x-1)^2} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x-1)}$$

$$= \lim_{x\to 1} \frac{x^2+x+1}{x-1}$$

the limit doesn't exist

$$\lim_{x\to 3} \frac{x^2+4x}{x+3}$$

$$= \lim_{x\to 3} \frac{x^2+4x}{x+3}$$

$$= \lim_{x\to 3} \frac{3^2+4(3)}{3+3}$$

$$= \frac{21}{6} = \frac{7}{2}$$

H.W

let :-

$$f(x) = \begin{cases} 3-x & , x<2 \\ 2 & , x=2 \\ \frac{x}{2} & , x>2 \end{cases}$$

- find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$

- Does $\lim_{x\to 2} f(x)$ exist ? why ?