Iterative Methods for Solving System of Linear Equation

7.1 Jacobi Method

Let the given equation be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

If the given system of equation is diagonally dominant then

$$x^{(i+1)} = \frac{1}{a_1}\left(d_1 - b_1y^{(i)} - c_1z^{(i)}\right)$$

$$y^{(i+1)} = \frac{1}{b_2}\left(d_2 - a_2x^{(i)} - c_2z^{(i)}\right)$$

$$z^{(i+1)} = \frac{1}{c_3}\left(d_3 - a_3x^{(i)} - b_3y^{(i)}\right)$$

7.1.1 Condition for Jacobi method of converges:

The sufficient condition is

$$|a_1| \geq |b_1| + |c_1|$$
$$|b_2| \geq |a_2| + |c_2|$$
$$|c_3| \geq |a_3| + |b_3|$$

The absolute value of the diagonal element in each row of the coefficient matrix must be greater than the sum of the absolute values of the off-diagonal elements in the same row.

Example 7.1:

Use the Jacobi iteration method to obtain the solution of the following equations:

$6x_1 - 2x_2 + x_3 = 11$

$x_1 + 2x_2 - 5x_3 = -1$

$-2x_1 + 7x_2 + 2x_3 = 5$

Solution

Step 1: Re-write the equations such that each equation has the unknown with largest coefficient on the left hand side:

$6x_1 = 11 + 2x_2 - x_3$

$7x_2 = 5 + 2x_1 - 2x_3$

$5x_3 = 1 + x_1 + 2x_2$

$$x_1 = \frac{2x_2 - x_3 + 11}{6}$$

$$x_2 = \frac{2x_1 - 2x_3 + 5}{7}$$

$$x_3 = \frac{x_1 + 2x_2 + 1}{5}$$

Step 2: Assume the initial guesses $x_1^0 = x_2^0 = x_3^0 = 0$ then calculate x_1^1, x_2^1 *and* x_3^1:

$$x_1^1 = \frac{2(x_2^0) - (x_3^0) + 11}{6} = \frac{2(0) - (0) + 11}{6} = 1.833$$

$$x_2^1 = \frac{2(x_1^0) - 2(x_3^0) + 5}{7} = \frac{2(0) - 2(0) + 5}{7} = 0.714$$

$$x_3^1 = \frac{(x_1^0) + 2(x_2^0) + 1}{5} = \frac{(0) + 2(0) + 1}{5} = 0.200$$

Step 3: Use the values obtained in the first iteration, to calculate the values for the 2nd iteration:

$$x_1^2 = \frac{2(x_2^1) - (x_3^1) + 11}{6} = \frac{2(0.714) - (0.200) + 11}{6} = 2.038$$

$$x_2^2 = \frac{2(x_1^1) - 2(x_3^1) + 5}{7} = \frac{2(1.833) - 2(0.200) + 5}{7} = 1.181$$

$$x_3^2 = \frac{(x_1^1) + 2(x_2^1) + 1}{5} = \frac{(1.833) + 2(0.714) + 1}{5} = 0.852$$

and so on for the next iterations so that the next values are calculated using the current values:

$$x_1^{i+1} = \frac{2(x_2^i) - (x_3^i) + 11}{6}$$

$$x_2^{i+1} = \frac{2(x_1^i) - 2(x_3^i) + 5}{7}$$

$$x_3^{i+1} = \frac{(x_1^i) + 2(x_2^i) + 1}{5}$$

The results for 9 iterations are:

	Unknowns		
Iter.	x_1	x_2	x_3
1	1.833	0.714	0.200
2	2.038	1.181	0.852
3	2.085	1.053	1.080
4	2.004	1.001	1.038
.	.	.	.
.	.	.	.
9	2.000	1.000	1.000

Example 7.2:

Solve the equations by Jacobi method

$$20x_1 + x_2 - 2x_3 = 17$$
$$3x_1 + 20x_2 - x_3 = -18$$
$$2x_1 - 3x_2 + 20x_3 = 25$$

Solution

Rewrite the given equation in the form:

$$x_1^{i+1} = \frac{1}{20}(17 - x_2^i + 2x_3^i)$$

$$x_2^{i+1} = \frac{1}{20}(-18 - 3x_1^i + x_3^i)$$

$$x_3^{i+1} = \frac{1}{20}(25 - 2x_1^i + 3x_2^i)$$

Using $x_1^0 = x_2^0 = x_3^0 = 0$, we obtain

$$x_1^1 = \frac{17}{20} = 0.85$$

$$x_2^1 = \frac{-18}{20} = -0.90$$

$$x_3^1 = \frac{25}{20} = 1.25$$

Putting these values on the right of equations to obtain

$$x_1^2 = \frac{1}{20}\left(17 - x_2^1 - 2x_3^1\right) = 1.02$$

$$x_2^2 = \frac{1}{20}\left(-18 - 3x_1^1 + x_3^1\right) = -0.965$$

$$x_3^2 = \frac{1}{20}\left(25 - 2x_1^1 + 3x_2^1\right) = 1.1515$$

These and further iterates are listed in the table below:

i	x_1^i	x_2^i	x_3^i
0	0	0	0
1	0.85	–0.90	1.25
2	1.02	–0.965	1.1515
3	1.0134	–0.9954	1.0032
4	1.0009	–1.0018	0.9993
5	1.0000	–1.0002	0.9996
6	1.0000	–1.0000	1.0000

The values in 5^{th} and 6^{th} iterations being practically the same, we can stop. Hence the solutions are:

$$x_1 = 1,\ x_2 = -1 \text{ and } x_3 = 1$$

7.2 Gauss-Seidel Method

If the given system of equation is diagonally dominant then

$$x^{(i+1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(i)} - c_1 z^{(i)}\right)$$

$$y^{(i+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(i+1)} - c_2 z^{(i)}\right)$$

$$z^{(i+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(i+1)} - b_3 y^{(i+1)}\right)$$

Example 7.3:

Use the Gauss-Seidel method to obtain the solution of the following equations:

$6x_1 - 2x_2 + x_3 = 11$ (1)

$x_1 + 2x_2 - 5x_3 = -1$ (2)

$-2x_1 + 7x_2 + 2x_3 = 5$ (3)

Solution

Step 1: Re-write the equations such that each equation has the unknown with largest `coefficient on the left hand side:

$$x_1 = \frac{2x_2 - x_3 + 11}{6} \quad \text{from eq. (1)}$$

$$x_2 = \frac{2x_1 - 2x_3 + 5}{7} \quad \text{from eq. (3)}$$

$$x_3 = \frac{x_1 + 2x_2 + 1}{5} \quad \text{from eq. (2)}$$

Step 2: Assume the initial guesses $x_2^0 = x_3^0 = 0$, then calculate x_1^1:

$$x_1^1 = \frac{2(x_2^0) - (x_3^0) + 11}{6} = \frac{2(0) - (0) + 11}{6} = 1.833$$

Use the updated value $x_1^1 = 1.833$ and $x_3^0 = 0$ to calculate x_2^1

$$x_2^1 = \frac{2(x_1^1) - 2(x_3^0) + 5}{7} = \frac{2(1.833) - 2(0) + 5}{7} = 1.238$$

Similarly, use $x_1^1 = 1.833$ and $x_2^1 = 1.238$ to calculate x_3^1

$$x_3^1 = \frac{(x_1^1) + 2(x_2^1) + 1}{5} = \frac{(1.833) + 2(1.238) + 1}{5} = 1.062$$

Step 3: Repeat the same procedure for the 2^{nd} iteration

$$x_1^2 = \frac{2(x_2^1) - (x_3^1) + 11}{6} = \frac{2(1.238) - (1.062) + 11}{6} = 2.069$$

$$x_2^2 = \frac{2(x_1^2) - 2(x_3^1) + 5}{7} = \frac{2(2.069) - 2(1.062) + 5}{7} = 1.002$$

$$x_3^2 = \frac{(x_1^2)+2(x_2^2)+1}{5} = \frac{(2.069)+2(1.002)+1}{5} = 1.015$$

and so on for the next iterations so that the next values are calculated using the current values:

$$x_1^{i+1} = \frac{2(x_2^i)-(x_3^i)+11}{6}$$

$$x_2^{i+1} = \frac{2(x_1^{i+1})-2(x_3^i)+5}{7}$$

$$x_3^{i+1} = \frac{(x_1^{i+1})+2(x_2^{i+1})+1}{5}$$

and continue the above iterative procedure until $[(x_k)^{i+1} - (x_k)^i]/(x_k)^{i+1} < \epsilon$ for i=1,2 and 3.

The procedure yields the exact solution after 5 iterations only:

	Unknown		
Iter.	x_1	x_2	x_3
1	1.833	1.238	1.062
2	2.069	1.002	1.015
3	1.998	0.995	0.998
4	1.999	1.000	1.000
5	2.000	1.000	1.000

Example 7.4:

Solve by Gauss – Seidel method, the equations:

$20x_1 + x_2 - 2x_3 = 17$

$3x_1 + 20x_2 - x_3 = -18$

$2x_1 - 3x_2 + 20x_3 = 25$

Solution

As before, we start with initial estimate $x_1^0 = x_2^0 = x_3^0 = 0$. We write the given equation in the form

$$x_1^{i+1} = \frac{1}{20}(17 - x_2^i + 2x_3^i)$$

$$x_2^{i+1} = \frac{1}{20}(-18 - 3x_1^{i+1} + x_3^i)$$

$$x_3^{i+1} = \frac{1}{20}(25 - 2x_1^{i+1} + 3x_2^{i+1})$$

These and further iterates are listed in the table below:

i	x_1^i	x_2^i	x_3^i
0	0	0	0
1	0.8500	–1.0275	1.0109
2	1.0025	–0.9998	0.9998
3	1.0000	–1.0000	1.0000

The value in the 2nd and 3rd iterations being particularly the same, we can stop. Hence the solutions is $x_1 = 1$, $x_2 = -1$ and $x_3 = 1$.

Example 7.5:

For the below figure calculate the values of the unknown flow rates F1, F2 and F3 by using Gauss-Seidel Method

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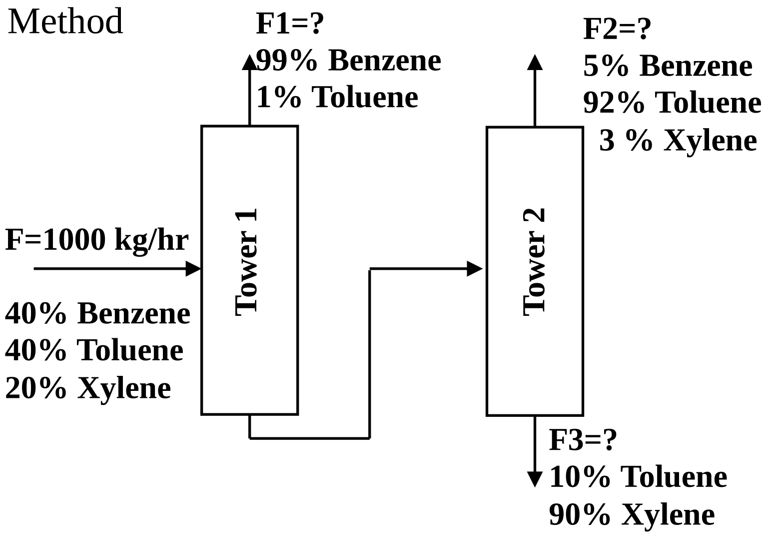

Component material balance gives these three equations of three variables

$$0.99F_1 + 0.05F_2 + 0F_3 = 400$$
$$0.01F_1 + 0.92F_2 + 0.1F_3 = 400$$
$$0F_1 + 0.03F_2 + 0.9F_3 = 200$$

Re-arranging the above equations

$$F_1 = (400 - 0.05F_2)/0.99$$
$$F_2 = (400 - 0.01F_1 - 0.1F_3)/0.92$$
$$F_3 = (200 - 0.03F_2)/0.9$$

Starting with F1=F2=F3=1000/3

Iteration	F1	F2	F3
	333.3333	333.3333	333.3333
1.0000	387.2054	394.3420	209.0775
2.0000	384.1241	407.8815	208.6262
3.0000	383.4403	407.9380	208.6243
4.0000	383.4375	407.9383	208.6243
5.0000	383.4375	407.9383	208.6243

A Matlab program for solving the above equations using Gauss-Seidel method is listed in Table 7.1

Table (7.1) Matlab code and results for solution example (7.5)

Matlab Code	**F1=333.33; F2=333.33; F3=333.33** **for i=1:4** **F1=(400-0.05*F2)/0.99;** **F2=(400-0.01*F1-0.1*F3)/0.92;** **F3=(200-0.03*F2)/0.9;** **disp([i, F1, F2, F3])** **end**
Results	**1.0000 387.2056 394.3423 209.0775** **2.0000 384.1241 407.8815 208.6262** **3.0000 383.4403 407.9380 208.6243** **4.0000 383.4375 407.9383 208.6243**