Application of Differentiation :

Def :-

1. a function $f(x)$ is increasing on (a,b) if for any $x_1, x_2 \in (a,b)$ and $x_1 < x_2$ then $f(x_1) < f(x_2)$.

2. a function $f(x)$ is decreasing on (a,b) if for any $x_1, x_2 \in (a,b)$ and $x_1 < x_2$ then $f(x_1) > f(x_2)$.

Theorem :-

1. If $f'(x) > 0$ on (a,b) then f is increase on (a,b).
2. If $f'(x) < 0$ on (a,b) then f is decrease on (a,b).

Ex: On which intervals is the function $f(x) = x^3 - 3x^2 + 1$ increase, decrease ?

Sol:

$f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x + 0$

$f'(x) = 0$

$3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

$x = 0 \,,\; x = 2$

$\xleftarrow{\quad + + + + + + \quad - - - - - - \quad + + + + + + + \quad}$ sign of $f(x)$

$\qquad\qquad 0 \qquad\qquad 2$

$f(x)$ is increase on $(-\infty, 0) \cup (2, \infty)$ and decrease on $(0, 2)$

Def : a function $f(x)$ is concave up on (a, b) if $f''(x) > 0$ and concave down on (a, b) if $f''(x) < 0$.

Ex : find the intervals where the fun. $f(x) = x^3 - 3x^2 + 1$ concave up, concave down ?

$f''(x) = 6x - 6$

$f''(x) = 0 \Rightarrow 6x - 6 = 0 \Rightarrow 6(x-1) = 0 \Rightarrow x = 1$

$\xleftarrow{\quad - - - - - - \quad + + + + + + + \quad}$ sign of $f''(x)$

$\qquad\qquad\quad 1$

concave up on $(1, \infty)$ and concave down $(-\infty, 1)$

- let f be a continuous function on $[a, b]$ and f changes direction of concavity $(x_0, f(x_0))$ called an inflection point of f if $f''(x) = 0$.

Ex: Find the location of all inflection points of $f(x) = x^4 - 8x^2 + 16$

Sol: $f'(x) = 4x^3 - 16x$

$f''(x) = 12x^2 - 16$

$12x^2 - 16 = 0$

$4(3x^2 - 4) = 0 \Rightarrow x = \pm \frac{2}{\sqrt{3}}$

$\left(\frac{2}{\sqrt{3}}, 7.1\right)$ and $\left(\frac{-2}{\sqrt{3}}, 7.1\right)$ are inflection points.

Ex: Find the intervals in which the function $f(x) = x^{1/3}(x+4)$ increasing, decreasing, concave up, concave down and inflection point ?

Sol:

$f(x) = x^{1/3}(x+4)$

$f'(x) = x^{1/3}(1) + (x+4) * \frac{1}{3} x^{-2/3}$

$f'(x) = x^{1/3} + \frac{x+4}{3x^{2/3}}$

$= \frac{4x+4}{3x^{2/3}} = \frac{4(x+1)}{3x^{2/3}}$

$- - - - - \quad (-1) \quad + + + \quad (0) \quad + + + + + +$

Increase on $(-1, 0)$ $(0, \infty)$

decrease on $(-1, -\infty)$

$$f'(x) = \frac{4}{3}(x+1)\, x^{-2/3}$$

$$f''(x) = \frac{4}{3}\left[(x+1)\left(-\frac{2}{3}\, x^{-5/3}\right) + x^{-2/3} * 1\right]$$

$$= \frac{4(x-2)}{9x^{5/3}} = 0 \quad \Rightarrow x = 2$$

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	0		2		sign of $f''(x)$

Concave up $(2,\infty)$, $(0,-\infty)$

Concave down $(0,2)$

Note : a function f is said to be have arelative extreme if it has either relative max. or relative min.

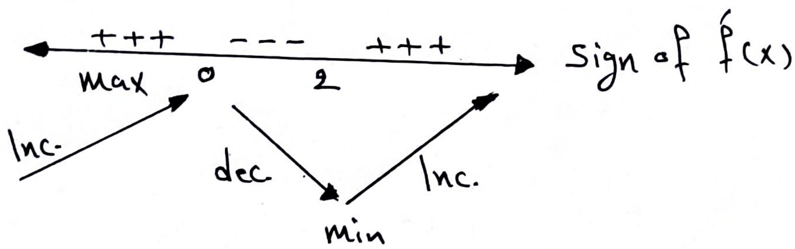
Def: acritical point for a function f is any value of x such that $f'(x) = 0$ or f is not differentiable and the critical point where $f'(x) = 0$ is called stationary point.

Ex: locate the relative extreme of $f(x) = 3x^{5/3} - 15x^{2/3}$?

Sol: $f'(x) = 5x^{2/3} - 10x^{-1/3}$

$$= 5x^{-1/3}(x-2) = \frac{5(x-2)}{x^{1/3}}$$

$f'(x) = 0 \Rightarrow \quad x = 0 \ , \ x = 2$

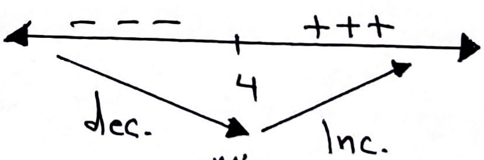


Ex: find all critical points, local min. and max. and inflection point $y = (x-4)^2$?

Sol: $y' = 2(x-4)(1) \Rightarrow 2x - 8 \Rightarrow y' = 0$

$2x - 8 = 0 \Rightarrow 2x = 8 \Rightarrow x = 4$

Sub 3 in $y' \Rightarrow y' < 0$
(decreasing)



Sub 5 in $y' \Rightarrow y' > 0$
(Increasing)

sub $x = 4$ in $y = (x-4)^2 \Rightarrow y = (4-4)^2 = 0$

$(4,0)$ Min. point

$y'' = 2 \Rightarrow y'' > 0$ Increasing - Concave up (Inflection point doesn't exist)