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3. Function of two or more variables

If (F) is a function of two independent variables we usually call the variables X and Y and the domain of (F) are a region in the xy-plane.

Domain:

Function	Domain	Range
$W=\sqrt{y-x}$	$y \geq x$	$w \geq 0$
$W=\frac{1}{xyz}$	$x,y\&z \neq 0$	$w \neq 0$
W=sin xy	X,Y real number	$-1 \leq w \leq 1$
$w=\sqrt{x^2+y^2+z^2}$	All real number	$w \geq 0$

Limits:

We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0), and write:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$

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Properties of limits of functions of two variables:

The following rules hold if L, M, k are real numbers and:

$$\lim_{(x,y)\to(x_o,y_o)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_o,y_o)} g(x,y) = M$$

1. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y)+g(x,y)) = L + M$

2. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y)-g(x,y)) = L - M$

3. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y).g(x,y)) = L.M$

4. $\lim_{(x,y)\to(x_o,y_o)} (kf(x,y)) = kL$ (any number of k)

5. $\lim_{(x,y)\to(x_o,y_o)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ $M \neq 0$

Example:

$$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y3} = \frac{0-(0)(1)+3}{(0)^2(1)+5(0)(1)-(1)^3} = \frac{3}{0+0-1} = -3$$

3.1 Partial derivatives

$f(x,y)$ (x_o,y_o)

$$\left.\frac{\partial f}{\partial x}\right|_{(x_o,y_o)} = \lim_{h\to 0}\frac{f(x_o+h,y_o)-f(x_o,y_o)}{h}$$

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$$f(x,y) \qquad (x_o, y_o)$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_o,y_o)} = \frac{d}{dy} f(x_o,y)\Big|_{y=y_o} = \lim_{h\to 0} \frac{f(x_o, y_o+h) - f(x_o,y_o)}{h}$$

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

Example 1: Find $\frac{\partial f}{\partial y}.\ \frac{\partial f}{\partial x}\ for\ f(x.y) = x^2 + 2xy + y^2 + 5$

Solution//

$$\frac{\partial f}{\partial y} = 0 + 2x + 2y + 0$$

$$\frac{\partial f}{\partial x} = 2x + 2y + 0 + 0$$

Example 2: Find $\frac{\partial f}{\partial y}.\ \frac{\partial f}{\partial x}\ for\ f(x.y) = y \sin xy$

Solution// $\frac{\partial f}{\partial y} = y \cos xy * x + \sin xy * 1 = xy \cos xy + \sin xy$

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Example **3: Find** $\frac{\partial f}{\partial z}$ $\boldsymbol{for\ f(x.y) = x\sin(y+3z)}$.

Solution //

$$\frac{\partial f}{\partial z} = 3x\cos(y+3z)$$

Exercise: Find $\frac{\partial f}{\partial y}.\ \frac{\partial f}{\partial x}$ $\boldsymbol{for\ f(x.y) = x^2 + y^2}$

Exercise: Find $\frac{\partial f}{\partial y}.\ \frac{\partial f}{\partial x}$ $\boldsymbol{for\ f(x.y) = \frac{\sin xy^2}{x^2+1}}$

Notes:-

The higher order partial derivative:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

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$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Example: Find f_{xx} ، f_{yy} ، f_{yx} and f_{xy} if $f(x,y) = x\cos y + ye^x$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \boxed{\cos y + ye^x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = \boxed{-x\sin y + e^x}$$

$\therefore$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos y + ye^x) = ye^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-x\sin y + e^x) = -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-x\sin y + e^x) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\cos y + ye^x) = -\sin y + e^x$$

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Exercise: Find $\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial^2 f}{\partial x \partial y}\, for$

$$f(x.y) = e^{x2\,y} + xy^3$$

Exercise: Find $\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial^2 f}{\partial x \partial y}\, for$

$$f(x.y) = xy + \frac{e^y}{y^2 + 2}$$

3.2 Chain rule

1-If Fx&Fy are contain &w=F(x,y),y&x are function of (t) only.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

2-If W is a function of (x,y&y) in the same time

x=F(r,s),y=F(r,s)&z=F(r,s)then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

ي**Example1: Find the derivative of w=3xy if x=cos t**

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&y=sin t

Solution//

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{dw}{dy} = 3x. \qquad \frac{\partial w}{\partial x} = 3y$$

$$\frac{\partial x}{\partial t} = -\sin t. \qquad \frac{\partial y}{\partial t} = \cos t$$

$$\frac{\partial w}{\partial t} = 3y * (-\sin t) + 3x * \cos t$$

$$\frac{\partial w}{\partial t} = -3 * \sin^2 t + 3\cos^2 t$$

Example2: Find the derivative of w=xy+z if x=cos t ,y=sin t &z=2t^2

Solution//

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$

=y.(-sin t)+x*cos t+1*4t = - $\sin^2 t$ +$\cos^2 t$ +4t

Exercise: Find $\frac{\partial w}{\partial t}$ *in terms of* $w = x^2 + y^2 . x =$

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$\cos t . y = \sin t$

Example 3: Find $\frac{\partial w}{\partial t}$ .w=2ye^x-ln z , x=ln(t^2+1) ,y=$\tan^{-1} t$ &z=e^t

Solution//

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial t} = 2y\, e^x . \frac{2t}{t^2+1} + 2e^x . \frac{1}{t^2+1} + \frac{-1}{z} . e^t$$

$$\frac{\partial w}{\partial t} = 2\tan^{-1} t \;\; e^{\ln(t^2+1)} . \frac{2t}{t^2+1} + 2e^{\ln(t^2+1)} . \frac{1}{t^2+1} - \frac{1}{e^t} . e^t$$

Example 4: Find $\frac{\partial w}{\partial r}$ & $\frac{\partial w}{\partial s}$. in terms of r&s if w=x+4y+z^2 , x =$\frac{r}{s}$,y=$2r^2 + \ln 2s$ &z=2r

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

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$$\frac{\partial w}{\partial r}=\frac{1}{s}+16r+8r=\frac{1}{s}+24\,r$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial s}=\frac{-1}{s^2}+\frac{4}{s}$$

Example 5: Find $\frac{\partial w}{\partial r}$ & $\frac{\partial w}{\partial s}$ $.\,in\ terms\ of\ r\&s\ if$ w=x^2 $+y^2$, x=r $+$ s , y=$r-s$

Solution//

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial r}=2x.\,1+2y=4r$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial s}=2x.\,1+2y.\,(-1)=2(r+s)-2(r-s)=4s$$