



Maclaurin Series (Power Series about $x=0$)

Let $f(x)$ be a function with derivatives of all orders at $x=0$ and exists.

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \quad \text{--- ①}$$

then by successive differentiations, we get :

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

$$f''(x) = 2a_2 + (3 \times 2)a_3x + (4 \times 3)a_4x^2 + \dots$$

$$f'''(x) = (3 \times 2)a_3 + (4 \times 3 \times 2)a_4x + \dots$$

⋮

⋮

Putting $x=0$ in each term above, yields :



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$$f(0) = a_0$$

$$f'(0) = a_1$$

$$f''(0) = 2a_2$$

$$f'''(0) = 3 \times 2 a_3$$

⋮

⋮

⋮

$$f^{(n)}(0) = \dots \dots \dots n! a_n \rightarrow a_n = f^{(n)}(0) / n!$$

by sub. the coefficients (a_1, a_2, \dots, a_n) in eq. (1), got:

$$\left[f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots \right] \quad (2)$$

the above eq. is a power series about $(x=0)$ and known as MacLaurin Series for function $f(x)$.



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Ex) Find the Maclaurin's Series for the function $f(x) = \sin(mx)$?

Sol) ∞ $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0)$

then;

$$f(x) = \sin mx \rightarrow f(0) = 0$$

$$f'(x) = m \cos mx \rightarrow f'(0) = m$$

$$f''(x) = -m^2 \sin mx \rightarrow f''(0) = 0$$

$$f'''(x) = -m^3 \cos mx \rightarrow f'''(0) = -m^3$$

$$f^{(4)}(x) = m^4 \sin mx \rightarrow f^{(4)}(0) = 0$$

$$f^{(5)}(x) = m^5 \cos mx \rightarrow f^{(5)}(0) = m^5$$



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by sub. in (2) above \rightarrow get $f(x) = \sin mx \rightarrow$

$$\sin mx = 0 + mx + \frac{x^2}{2!} (0) + \frac{x^3}{3!} (-m^3) + \frac{x^4}{4!} (0) + \frac{x^5}{5!} m^5 + \dots$$

$$\sin mx = mx - m^3 \frac{x^3}{3!} + m^5 \frac{x^5}{5!} - m^7 \frac{x^7}{7!} + m^9 \frac{x^9}{9!} - \dots$$

Now putting $(m=1) \rightarrow$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$



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Ex) Find Maclaurin's Series for $f(x) = \ln(1+x)$?

Sol.) $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$

$$f(x) = \ln(1+x) \rightarrow f(0) = \ln(1+0) = 0$$

$$f'(x) = 1/(1+x) \rightarrow f'(0) = 1$$

$$f''(x) = (-1)/(1+x)^2 \rightarrow f''(0) = -1$$

$$f'''(x) = (2)/(1+x)^3 \rightarrow f'''(0) = 2$$

$$f^{(4)}(x) = (-6)/(1+x)^4 \rightarrow f^{(4)}(0) = -6 \rightarrow$$

$$\ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n+1}}{n} x^n + \dots \rightarrow$$



Ex) Find Maclaurin's series for $f(x) = e^{ax}$, then compute (e^2) ?

Sol.)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$

$$f(x) = e^{ax} \rightarrow f(0) = 1$$

$$f'(x) = a e^{ax} \rightarrow f'(0) = a$$

$$f''(x) = a^2 e^{ax} \rightarrow f''(0) = a^2$$

$$f'''(x) = a^3 e^{ax} \rightarrow f'''(0) = a^3$$

⋮

$$f^{(n)}(x) = a^n e^{ax} \rightarrow f^{(n)}(0) = a^n \rightarrow$$

$$e^{ax} = 1 + ax + a^2 \frac{x^2}{2!} + a^3 \frac{x^3}{3!} + \dots + a^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!}$$

Putting $a=1 \rightarrow$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

\rightarrow Putting $x=2 \rightarrow$ for example take up $(n=6)$

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} = 7.355555$$

$$e^2 (\text{calculator}) = 7.389$$