FOURIER TRANSFORM

To represent non-periodic signals in the frequency domain, we introduce the Fourier transform. Let $x(t)$ be a non-periodic signal. Then the *Fourier transform* of $x(t)$, symbolized by $\mathcal{F}$, is defined by

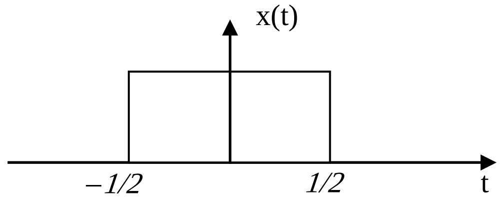
$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad (5.1)$$

The *inverse* Fourier transform of $X(\omega)$, symbolized by $\mathcal{F}^{-1}$, is defined by

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \qquad (5.2)$$

Example 3.1: Find the Fourier transform of the rectangular pulse signal x(t)defined by

Sol, $$x(t) = rect(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| > 1/2 \end{cases}$$



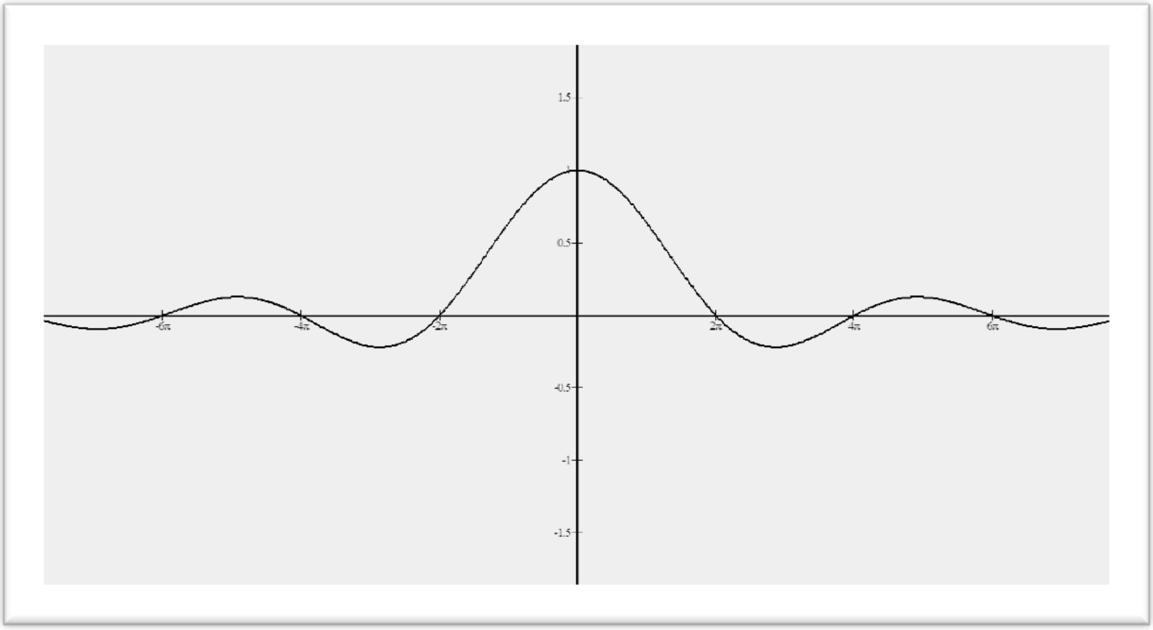
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-1/2}^{1/2} e^{-j\omega t}dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-1/2}^{1/2} = \frac{-1}{j\omega}\left[e^{-j\omega/2} - e^{j\omega/2}\right]$$

$$= \frac{1}{j\omega}\left[e^{j\omega/2} - e^{-j\omega/2}\right]$$

$$= \frac{2}{\omega}\left[\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j}\right]$$

$$= \frac{2}{\omega}\sin(\omega/2) = \frac{\sin(\omega/2)}{(\omega/2)} = \mathrm{sa}(\omega/2)$$



<u>3.1 Properties of the Fourier Transform</u>

We use the notation $x(t) \leftrightarrow X(\omega)$ to denote the Fourier transform pair.

<u>*3.1.1 Linearity (Superposition)*</u>

$$a_1x_1(t) \pm a_2x_2(t) \leftrightarrow a_1X_1(\omega) \pm a_2X_2(\omega) \qquad (5.3)$$

where a_1 and a_2 are any constants.

<u>*3.1.2 Time Shifting*</u>

$$x(t \pm t_o) \leftrightarrow X(\omega)e^{\pm j\omega t_o} \qquad (5.4)$$

For example we have in example1 $\mathcal{F}[rect(t)] = \mathrm{sa}(\omega/2)$, then

$\mathcal{F}[rect(t-2)] = e^{-2j\omega}\,\mathrm{sa}(\omega/2)$ ***(proof)***

<u>*3.1.3 Frequency shifting*</u>

$$x(t)e^{\pm j\omega_o t} \leftrightarrow X(\omega \mp \omega_o) \qquad (5.5)$$

For example we have in example1 $\mathcal{F}[rect(t)]$ = sa($\omega/2$), then

$\mathcal{F}\left[e^{2jt}rect(t)\right]$ = sa($\frac{\omega-2}{2}$)   ***(proof)***

3.1.4 Scaling

$$x(at)\leftrightarrow\frac{1}{|a|}X\left(\frac{\omega}{a}\right) \qquad (5.6)$$

For example we have in example1 $\mathcal{F}[rect(t)]$ = sa($\omega/2$), then

$\mathcal{F}[rect(2t)]$ = $\frac{1}{|2|}$ sa($\frac{\omega/2}{2}$)   ***(proof)***

3.1.5 Time differentiation

$$\frac{d}{dt}x(t)\leftrightarrow j\omega X(\omega) \qquad (5.7)$$

3.1.6 Time integration

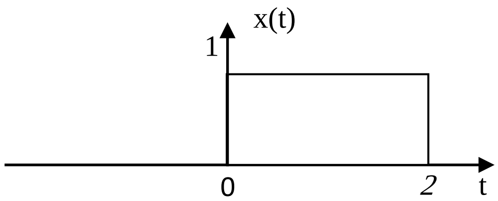
$$\int x(t)\leftrightarrow\frac{1}{j\omega}X(\omega) \qquad (5.8)$$

H.W: Verify Eqs. (3.3) to (3.8).

Example 3.2: Find the Fourier transform of the rectangular pulse signal *x*(t)defined by

$$x(t) = rect\left(\frac{t-1}{2}\right) = \begin{cases}1 & 0 < t < 2\\ 0 & elsewhere\end{cases}$$

Sol.



There are two solutions:

First solution: *Using the properties*

1) Applying scaling property

   we have in example1 $\mathcal{F}[rect(t)]$ = sa($\omega/2$), then

$$\mathcal{F}\left[rect\left(\frac{1}{2}t\right)\right] = \frac{1}{|1/2|}\,\text{sa}\left(\frac{\frac{\omega}{1/2}}{2}\right) = 2\ \text{sa}(\omega)$$

2) Applying time- shifting property

   we have $\mathcal{F}\left[rect\left(\frac{1}{2}t\right)\right]$ = 2 sa(ω), then

$$\mathcal{F}\left[rect\left(\frac{t-1}{2}\right)\right] = 2e^{-j\omega}\ \text{sa}(\omega)$$

Second solution: *Using the general equation*

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{0}^{2} 1.\, e^{-j\omega t}dt$$

$$= -\frac{1}{j\omega}\, e^{-j\omega t}\Big|_0^2$$

$$= \frac{-1}{j\omega}\left[e^{-j\omega * 2} - 1\right]$$

$$= \frac{1}{j\omega}\left[1 - e^{-j\omega * 2}\right]$$

$$= \frac{1}{j\omega}e^{-j\omega}\left[e^{j\omega} - e^{-j\omega}\right] \times\frac{2}{2}$$

$$= \frac{2}{\omega}\, e^{-j\omega}\left[\frac{e^{j\omega}-e^{-j\omega}}{2j}\right]$$

$$= \frac{2}{\omega}e^{-j\omega}\ \sin(\omega) = 2e^{-j\omega}\ \frac{\sin(\omega)}{(\omega)} = 2e^{-j\omega}\ \text{sa}(\omega)$$

3.2 Fourier Transforms of Some Useful Signals

$$\delta(t) \longleftrightarrow 1$$

$$\delta(t-t_o) \longleftrightarrow e^{-j\omega t_o}$$

$$1 \longleftrightarrow 2\pi\,\delta(\omega)$$

$$e^{j\omega t_o} \longleftrightarrow 2\pi\,\delta(\omega-\omega_o)$$

H.W// Find the F.T of the following signals:

(1) x(t) =A rect($\frac{t-3}{6}$),

(2) x(t)=Arect($\frac{t+2}{3}$),

(3)x(t)=cosω_0t

(4)x(t) =sgn(t)