Lecture 6
Proof methods

We will discuss four proof methods:

1. Direct proof
2. Indirect proof
3. Proof by contradiction
4. Counter examples

1. Direct proof:

Consider an implication: $p \rightarrow q$

- If p is false, then the implication is always true
- Thus, show that if p is true, then q is true

To perform a direct proof, assume that p is true, and show that q must therefore be true.

Example:

Show that the square of an even number is an even number?

Rephrased: if n is even, then n^2 is even

Solution:

- Assume n is even
- Thus, n = 2k, for some k (definition of even numbers)
- $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
- As n^2 is 2 times an integer, n^2 is thus even

2. Indirect proofs:

Consider an implication: $p \rightarrow q$

It's contrapositive is $\neg q \rightarrow \neg p$

It is logically equivalent to the original implication.

To proof that we must show that if $\neg q$ is true, then $\neg p$ is true.

To perform an indirect proof, do a direct proof on the contrapositive.

Example:

If n^2 is an odd integer then n is an odd integer

Prove the contrapositive: If n is an even integer, then n^2 is an even integer.
Proof: n=2k for some integer k (definition of even numbers).
$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
Since n^2 is 2 times an integer, it is even.

3. Proof by contradiction:

Given a statement *p*, assume it is false.

Assume $\neg p$.

Prove that $\neg p$ cannot occur.

- A contradiction exists.

Given a statement of the form $p \rightarrow q$.

- To assume it's false, you only have to consider the case where *p* is true and *q* is false.

Example 1

Theorem

There are infinitely many prime numbers.

Proof.

Assume there are a finite number of primes.

List them as follows: $p_1, p_2 \ldots, p_n$.

Consider the number $q = p_1 p_2 \ldots p_n + 1$

- This number is not divisible by any of the listed primes.

-If we divided p_i into q, there would result a remainder of 1

- We must conclude that q is a prime number, not among the primes listed above.

  -This contradicts our assumption that all primes are in the list $p_1, p_2 \ldots, p_n$.

Example 2

Prove that if n is an integer and n^3+5 is odd, then n is even.

Rephrased: If n^3+5 is odd, then n is even.

Proof

Assume p is true and q is false and show a contradiction.
That is if we assume p *and* $\neg q$,we can show that implies q .
The contradiction is q *and* $\neg q$.
We have follow the following steps:

Assume that n^3+5 is odd, and n is odd.

$n=2k+1$ for some integer k (definition of odd numbers)
$n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$
As $2(4k^3+6k^2+3k+3)$ is 2 times an integer, it must be even
Contradiction!

4.Counterexamples:

Every positive integer is the square of another integer.

Solution:

The square root of 5 is 2.236, which is not an integer.