Convergent and Divergent series

Geometric Series and P-Series Test

The tests used to determine the behavior of a Geometric and P-series follow a specific equation format. A Geometric Series is the sum of a set of terms, where each term, aa_{nn}, is being multiplied by some ratio, rr_{nn}. The Geometric Series Test compares rr with 1 to determine its behavior. A P-series is the sum of a set of terms, where the denominator of each term, 1_{n}, is raised to some pp value. Similarly, the P-series Test compares pp with 1 to determin its behavior.

Geometric Series Test $$\sum_{n=1}^{\infty} a_1 \,(r\,)^n$$	P-Series Test $$\sum_{n=1}^{\infty} \frac{1}{p^n}$$
Diverges: For $\|r\|\geq 1$ Converges: For $\|r\| < 1$; Converges to $a1/\ 1-r$	Diverges: For $0 < p \leq 1$ Converges: For $p > 1$

Steps to apply:

Step 1: Determine the type of series given.
Step 2: Determine the value of rr or pp based on the type of series.
Step 3: Use the appropriate condition to determine its behavior.
Step 4: If it is a converging Geometric Series, use $a1/1-r$ to find what it converges to

Example A: Determine if the series converges or diverges. If it converges, determine where the series converges.

$$\sum_{n=1}^{\infty} 7\left(\frac{3}{8}\right)^{n-1}$$

Step 1: Determine the type of series given.
The series depicts a number to the power of some nn variable. Therefore, this is considered a Geometric Series.
Step 2: Determine the value of rr or pp based on the type of series.
In this case, $\mathrm{r}=38$ and $\mathrm{a}1=7$.
Step 3: Use the appropriate condition to determine its behavior.
Based on the condition, $|\mathrm{r}|<1$, the given series must converge.
Step 4: If it is a converging Geometric Series, use $\mathrm{a}11-\mathrm{r}$ to find what it converges to.
In this case, $\mathrm{a}1/1-\mathrm{r} \rightarrow (7)/1$-$(\frac{3}{8})$. As a result, the given series would converge to $56/5$.

Example B: Determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{5}{\sqrt[2]{n^9}}$$

Step 1: Determine the type of series given.
The formula of a P-series is applicable if the numerator is 1.
Since the numerator is a constant, it can be factored out of the series:

$$5\sum_{n=1}^{\infty}\frac{1}{\sqrt[2]{n^9}}$$

Therefore, this series is a P-series.

Step 2: Determine the value of r or p based on the type of series.

The denominator, $\sqrt[2]{n^9}$, can be rewritten as n92; therefore, p=9/2.

Step 3 Use the appropriate condition to determine its behavior :

Based on the condition, p>1, this series converges.
Using the P-series Test where, p=92, it can be determined that the given series converges.

Divergence Test

If the given series cannot be compared to a Geometric or P-series, then the Divergence Test should be used. During this test, there will be times where L'Hopital's Rule (LHR) will be applied when the limit is ∞/∞ or $0/0$.

Divergence Test

Given $\sum_{n=1}^{\infty} a_n = 1$:

Diverges: If $\lim_{n\to\infty} a_n \neq 0$

Note: If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive. A different test should be used.

Steps to apply:
Step 1: Find the limit as nn approaches infinity.
Step 2: Determine if the limit satisfies the test condition.

Example A: Determined if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{7n^3 + 2n}{9 + 2n^2}$$

Step 1: Find the limit as *n* approaches infinity. $\sum_{n=1}^{n} \frac{7n^3+2n}{9+2n^2} \rightarrow \lim_{n\to\infty} \frac{7n^3+2n}{9+2n^2}$

To find the limit, start by plugging in ∞ for n: limn→∞(7(∞)3+2(∞)/9+2(∞)2)

For more information about computing limits, refer to ACE's Limit Handout.

Because the limit is ∞∞ , LHR is applied. LHR

limx→∞(f(x)/g(x))=limx→∞(f′(x)/g′(x))

In this case, f(n)/g(n)=7n3+2n/9+2n2 and through LHR this results in:

f′(n)/g′(n)=21n2+24n

Plugging in ∞ for n: limn→∞(21(∞)2+24(∞)), this results in ∞/∞ thus LHR is applied again.

In this case, f(n)/g(n)=21n2+24n and through LHR this results in:

f′(n)/g′(n)=42n/4

Plugging in ∞ for n: limn→∞42(∞)/4, this evaluates to ∞.

Step 2: Determine if the limit satisfies the test condition.
Since the limn→∞(7n3+2n/9+2n2)≠0, the series diverges

Integral Test

If the Divergence Test proves to be inconclusive, then the Integral Test should be performed. As the name suggests, this test will require the use of integration.

Integral Test
Given a series:

$\sum_{n=1}^{\infty} a_n$

All n will be replaced with x.

If $\int_1^{\infty} fx\,\mathrm{dx}$ converges, then the series converges.

If $\int_1^{\infty} fx\,\mathrm{dx}$ diverges, then the series diverges

Converges : $-\infty < \int_1^{\infty} f(x)dx < \infty$
Diverge : $\int_1^{\infty} f(x)dx = \pm\infty$

Note: The number evaluated by the integral is not where the series converges to. [Refer to image to the right.]

Steps to apply:
Step 1: Replace all n with xx.
Step 2: Integrate between 1 and ∞.

Step 3: Determine the series behavior based on test conditions.

Example A: Determine if the series converges or diverges.

$$\textbf{\textit{Step}}\ \mathbf{1}: Replace\ all\ n\ with\ x.$$

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^3} \rightarrow \int_1^{\infty} \frac{x}{(x^2+1)}\, dx$$

Step 2: Integrate between 1 and ∞.
Integrate the definite integral. This will require the use of u-substitution.
For more information on integration, refer to ACE's Integral Handout.

$$u=(x^2+1)\ ,\ du=2x\ dx$$

$$\frac{1}{2}\int_1^{\infty} \frac{1}{(x^2+1)^3}\, 2x\ dx \rightarrow 1/2\int_1^{\infty} \frac{1}{u^3}\, du \rightarrow \frac{1}{2}\ \int_1^{\infty} u^{-3}\, du$$

$$\frac{1}{2}\int_1^{\infty} u^{-3+1} \rightarrow \frac{1}{2}\int_1^{\infty} \frac{-1}{2} * u^{-2} \rightarrow -\frac{1}{4}\ u^{-2}|_1^{\infty} \rightarrow -\frac{1}{4}\left[\frac{1}{u^2}|_1^{\infty}\right]$$

$$-\frac{1}{4}\left[\frac{1}{(x^2+1)^2}|_1^{\infty}\right] \rightarrow -\frac{1}{4}\left[0-\frac{1}{4}\right] = \frac{1}{16}$$

Step 3: Determine the series behavior based on test conditions.
The integral evaluates to $1/16$ which is between $-\infty$ and ∞, thus the integral converges. Since the integral converges, the series must also converge.

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