Cauhy -Riemann Equation

Let f (z) =u(x,y) +vi (x,y), the necessary condition that f(z) be intuition in region Rishat u (x,y)sqtisfyi

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{———} \quad 1$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \quad \text{———} \quad 2$$

Ex/ check the following function if they are analytic.

1) f (z) =x -iy

u+vi =x-iy

u = x

v = - y

$$\frac{\partial u}{\partial x} = 1 \quad , \frac{\partial v}{\partial y} = -1$$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ $then\ f(z) is\ not\ analytic$

2) $f(z) = z^2 + 2z - 1$

$u + vi = (x + yi)^2 + 2(x + yi) - 1$

$u + vi = x^2 + 2xyi - y^2 + 2x + 2yi - 1$

$u + vi = x^2 - y^2 + 2x + 2xyi + 2yi - 1$

$u = x^2 - y^2 + 2x - 1$

$v = 2xy + 2y$

$\frac{\partial u}{\partial x} = 2x + 2 \qquad , \frac{\partial u}{\partial y} = 2x + 2$

$\frac{\partial u}{\partial y} = -2y \qquad , \frac{\partial u}{\partial x} = -2y$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial u}{\partial x}$$

Then f (z) is analytic

H . W

Check the following functions if analytic

1 – f (z) = z

2 – f (z) = z+z

3 – f (z) = e^z

4 – f (z) = $\frac{1}{z}$

Harmonic function

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$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Ex/ prof that u (x , y) = $e^{y}\cos x$ is harmonic

$\acute{u}_x = e^{y}\sin x \qquad \bar{\bar{u}}_{xx} = -e^{y}\cos x$

$\acute{u}_y = e^{y}\cos x \qquad \bar{\bar{u}}_{yy} = e^{y}\cos x$

$\bar{\bar{u}}_{xx} + \bar{\bar{u}}_{yy} = 0$

$-e^{y}\cos x + e^{y}\cos x = 0$

2 – u (x, y) = ln (x^2+y^2)

$$\acute{u}_x = \frac{2x}{x^2+y^2} \quad , \quad \bar{\bar{u}}_{xx} = \frac{2(x^2+y^2)-2\,(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x}{(x^2+y^2)^2}$$

$$\acute{u}_y = \frac{2y}{x^2+y^2} \qquad \bar{\bar{u}}_{yy} = \frac{2(x^2+y^2)-2y\,(2y)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}$$

$\bar{\bar{u}}_{xx} + \bar{\bar{u}}_{yy} = 0$

Ex/Find harmonic f (z)that real part is

u (x ,y) = sin hx sin y. and harmonic conjugate?

$\frac{\partial u}{\partial x}$ = cos hx sin y

$\frac{\partial u}{\partial^2 x}$ = sin hx sin y

$\frac{\partial u}{\partial y}$ =-sin hx cos y

$\frac{\partial u}{\partial^2 y}$ =-sin hx sin y

$\frac{\partial u}{\partial^2 x} + \frac{\partial u}{\partial^2 y}$ = 0

$\frac{\partial u}{\partial x^2} = \frac{\partial v}{\partial y}$

cos hx sin y $\partial_y = \partial_v$

$\int dv$ = cos hx $\int sin$ y dy

v =-cos hx cos y + $\varphi (x)$

$\frac{\partial v}{\partial x}$ -sin x cos y + $\acute{\varphi} (x)$

$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

Sin hx cos y = sin hx cosy $\acute{\varphi}\,(x)$

$\int \varphi\,(x) = \int \theta$ ⟶ $\varphi\,(x)$ ⟶ sub in (1)

v = -cos hx cos y + c

f (z) = u + vi

H.W /Prof that u (x , y) = y3-3xy2 harmonic function and find harmonic conjugate.

Double Integration

The definite integral can be extended to functions of more than one variable. Consider, for example, a function of two variables z = f (x, y). The double integral of function f (x, y) is denoted by

$$\iint_R F(x,y)\,dA$$

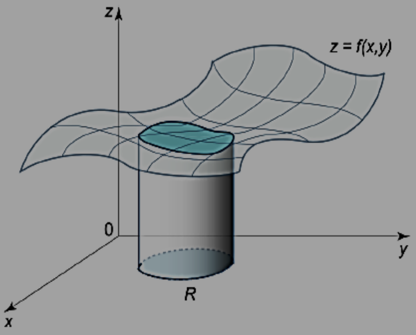


Figure 1

Where R is the region of integration in the xy-plane.

Properties of double integral

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\,dA = \iint_R f(x, y)\,dA \pm \iint_R g(x, y)\,dA$$

3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\,dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x, y)\,dA \geq \iint_R g(x, y)\,dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\displaystyle\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$

if R is the union of two nonoverlapping regions R_1 and R_2

Example 1//Evaluate $\int_0^3 \int_0^2 (4-y^2)\, dy\, dx$

Solution //

$$\int_0^3 \int_0^2 (4-y^2)\, dy\, dx = \int_0^3 (4y - \frac{y^3}{3})\Big|_0^2 dx$$

$$= \int_0^3 \frac{16}{3}\, dx = (\frac{16}{3}x)\Big|_0^3 = 16$$

Example 2 //Evaluate $\int_\pi^{2\pi} \int_0^\pi (sinx + cosy)\, dx\, dy$

Solution//

$$\int_\pi^{2\pi} \int_0^\pi (sinx + cosy)\, dx\, dy = \int_0^\pi (-cosx + xcosy)\Big|_0^\pi dy$$

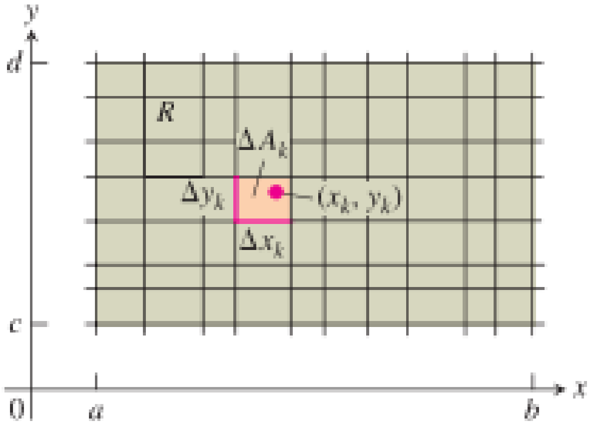
$$= \int_0^\pi ((1+\pi\, cosy) - (-1+0))dy$$

$$= \int_0^\pi (2+\pi cosy)\, dy = (2y + \pi\, siny)\Big|_\pi^{2\pi} = 2\pi$$

Finding Limits of Integration in cartesian form:

-Area

$$A = \iint dx\, dy \quad or \quad A = \iint dy\, dx$$



Example //Find area enclosed by x=3, x=1, y=0, y=2. using double integration.

Solution/ /

A=$\iint dx\,dy$

A=$\int_0^2 \int_1^3 dx\,dy = \int_0^2 (x)|_1^3 dy = \int_0^2 2\,dy = (2y)|_0^2 = 4$

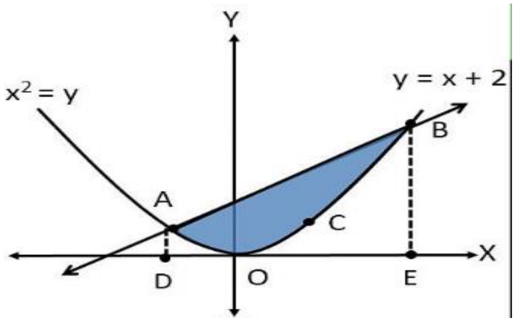
Or

A=$\iint dy\,dx$

A=$\int_1^3 \int_0^2 dy\,dx = \int_1^3 (y)|_0^2 dx = \int_1^3 2\,dx = (2x)|_1^3 = 4$

Example //Find the area enclosed by y=x+2 & .
y=x^2

Solution //



$$y_1 = y_2$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(X+1)(x-2) = 0$$

$$X = -1,\ x = 2$$

$$A = \iint dy\, dx$$

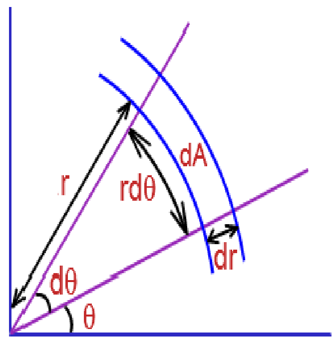
$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx = \int_{-1}^{2} (y) \Big|_{x^2}^{x+2} dx = \int_{-1}^{2} (x+2) - x^2)\, dx = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right) \Big|_{-1}^{2} = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 4.5$$

Finding Limits of Integration in polar form

$$\iint_R F(r,\theta)\, dA = \int_{r_1}^{r_2} \int_{\theta_1 = g_1(r)}^{\theta_2 = g_2(r)} F(r,\theta)\ r\, d\theta\, dr$$

or

$$\iint_R F(r,\theta)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1 = g_1(\theta)}^{r_2 = g_2(\theta)} F(r,\theta)\ r\, dr\, d\theta$$



-Area

$$A = \iint r\, dr\, d\theta$$

Example //Find the area enclosed by lemniscate r^2=4cos2θ

Solution//

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4cos2\theta}} r\,dr\,d\Theta = \int_0^{\pi/4} \left(\frac{r^2}{2}\right)\Big|_0^{\sqrt{4cos2\theta}} d\Theta$$

$$= \int_0^{\pi/4} 2\cos2\Theta dx = 4sin2\Theta\Big|_0^{\pi/4} = 4$$

Triple integral

If f (x, y, z) is a function defined on a closed bounded region D in space, such as the region occupied by a solid ball or a lump of clay, then the integral of f over D may be defined in the following way.

$$V = \iiint_D dV = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$

Example 1//Evaluate $\int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\,dz\,dy\,dx$

Solution//

$$\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^1 \left(x^2 z + y^2 z + \frac{z^3}{3}\right)\Big|_0^1 dydx$$

$$= \int_0^1 \int_0^1 \left(x^2 + y^2 + \frac{1}{3}\right) dydx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3} + \frac{1}{3}y\right)\Big|_0^1 dx = \int_0^1 \left(x^2 + \frac{2}{3}\right) dx$$

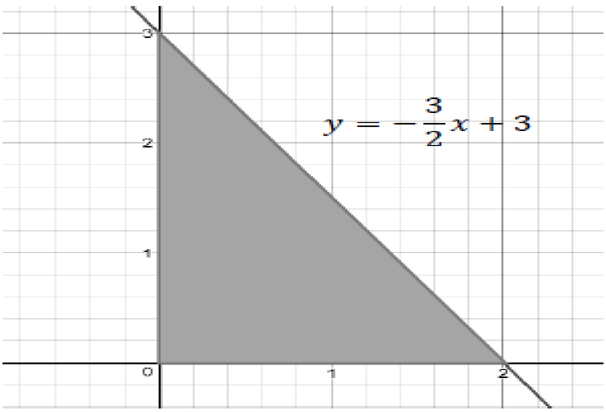
$$= \left(\frac{x^3}{3} + \frac{2}{3}x\right)\Big|_0^1 = 1$$

Surface area

$$[S=\int_a^b \int_a^b \sqrt{1 + (\frac{df}{dx})^2 + (\frac{df}{dy})^2} dy\, dx]$$

Example// Find the area of the following surfaces:

z=f(x, y)=6-3-2y lies in the region shown in fig.



Solution /

$$\frac{\partial f}{\partial x} = -3 \quad , \quad \frac{\partial f}{\partial y} = -2 \quad , \quad 0 \le x \le 2 \quad , \quad 0 \le y \le -\frac{3}{2}x + 3$$

$$S = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

$$= \int_0^2 \int_0^{-\frac{3}{2}x+3} \sqrt{(-3)^2 + (-2)^2 + 1}\, dy\, dx = \sqrt{14}\int_0^2 \left(-\frac{3}{2}x + 3\right) dx$$

$$S = \sqrt{14}\left(-\frac{3}{4}x^2 + 3x\right)\Bigg|_0^2 = 3\sqrt{14}$$