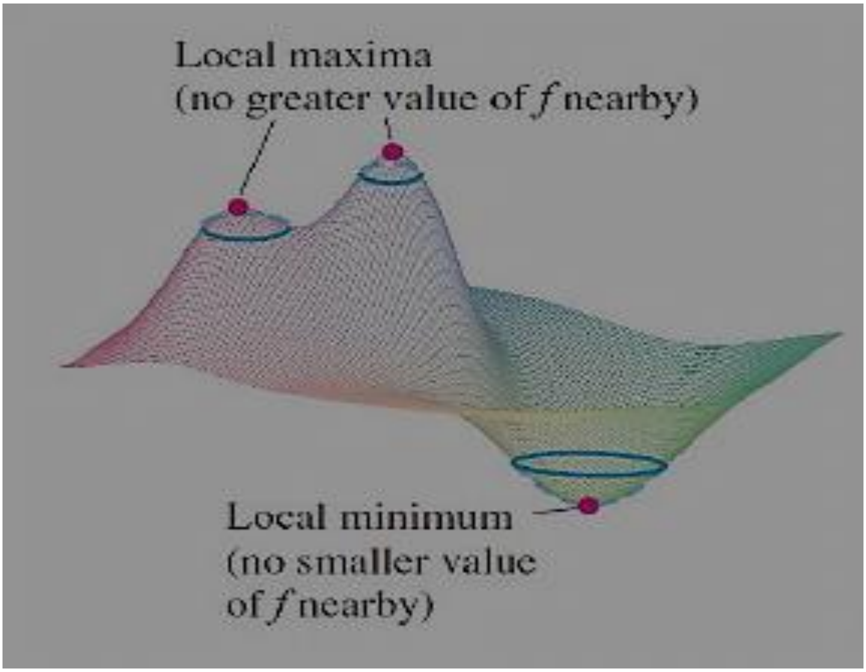
3.14 The extremes(max,min &saddle points)

To find the local extreme values of a function of a single, we look for points where the graph has a horizontal tangent line. At such points, we then look for **local maxima**, **local minima**, and points of inflection. For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a horizontal tangent plane. At such points, we then look for **local maxima**, **local minima**, and **saddle points** (more about saddle points

in a moment)



Then:

1. if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^{\ 2} > 0$ at $(a,b) \Longrightarrow$ then f has **a local maximum** at (a,b)

2. if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^{\ 2} > 0$ at $(a,b) \Longrightarrow$ then f has **a local minimum** at (a,b)

3. if $f_{xx}f_{yy} - f_{xy}^{\ 2} < 0$ at $(a,b) \Longrightarrow$ then f has **a saddle point** at (a,b)

4. if $f_{xx}f_{yy} - f_{xy}^{\ 2} = 0$ at $(a,b) \Longrightarrow$ then **the test inconclusive** at (a,b).

In this case we must find some other way to determine the behavior of f at (a,b)

<u>Note</u>: -

$f_x = 0$ and $f_y = 0 \Longrightarrow$ solve these equation to find the value of $(x,y) = (a,b) \Longrightarrow$ (*critical point*)

Example: find the extreme values of the function

$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy - x^2 - y^2 - 2x - 2y + 4) = \boxed{y - 2x - 2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy - x^2 - y^2 - 2x - 2y + 4) = \boxed{x - 2y - 2}$$

$$\left.\begin{array}{l} f_x = 0 \implies y - 2x - 2 = 0 \\ f_y = 0 \implies x - 2y - 2 = 0 \end{array}\right\} \implies$$

Solve these equation to find $(x, y) \implies (a, b)$

$$\left.\begin{array}{l} x = -2 \implies a = -2 \\ y = -2 \implies b = -2 \end{array}\right\} \implies$$

Critical point $(-2,-2)$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(y - 2x - 2) = -2$$

$$\therefore\ f_{xx}(-2,-2) = \boxed{-2}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x - 2y - 2) = -2$$

$$\therefore\ f_{yy}(-2,-2) = \boxed{-2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x - 2y - 2) = 1$$

$$f_{xx}f_{yy} - f_{xy}^{\ 2} = (-2)(-2) - (1)^2 = 4 - 1 = \boxed{3}$$

$$f_{xx} < 0 \ \text{ and } \ f_{xx}f_{yy} - f_{xy}^{\ 2} > 0 \implies \therefore f \text{ has a local maximum at } (-2,-2)$$

The value of f at this point is:

$$f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - (2)(-2) - (2)(-2) + 4$$
$$= 4 - 4 - 4 + 4 + 4 + 4 = \boxed{8}$$

Example: find the local maxima, local minima, and saddle point of the function

$$f(x,y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$$

Solution:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + 3y^2 - 6x + 3y - 6) = \boxed{2x + 3y - 6}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + 3y^2 - 6x + 3y - 6) = \boxed{3x + 6y + 3}$$

$$\left.\begin{array}{l} f_x = 0 \implies 2x + 3y - 6 = 0 \\ f_y = 0 \implies 3x + 6y + 3 = 0 \end{array}\right\} \implies \text{Solve these equation to find } (x, y) \implies (a, b)$$

$$\left.\begin{array}{l} x = 15 \implies a = 15 \\ y = -8 \implies b = -8 \end{array}\right\} \implies \text{Critical point } (15,-8)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + 3y - 6) = 2$$

$$\therefore\ f_{xx}(15,-8) = \boxed{2}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x + 6y + 3) = 6$$

$$\therefore\ f_{yy}(15,-8) = \boxed{6}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x + 6y + 3) = 3$$

$$f_{xx}f_{yy} - f_{xy}^{\ 2} = (2)(6) - (3)^2 = 12 - 9 = \boxed{3}$$

$f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^{\ 2} > 0 \implies \therefore f$ has a local minimum at $(15,-8)$

The value of f at this point is:

$$f(15,-8) = (15)^2 + (3(15)(-8) + (3)(-8)^2 - (6)(15) + (3)(-8) - 6$$

$$= 225 - 360 + 192 - 90 - 24 - 6 = \boxed{-63}$$