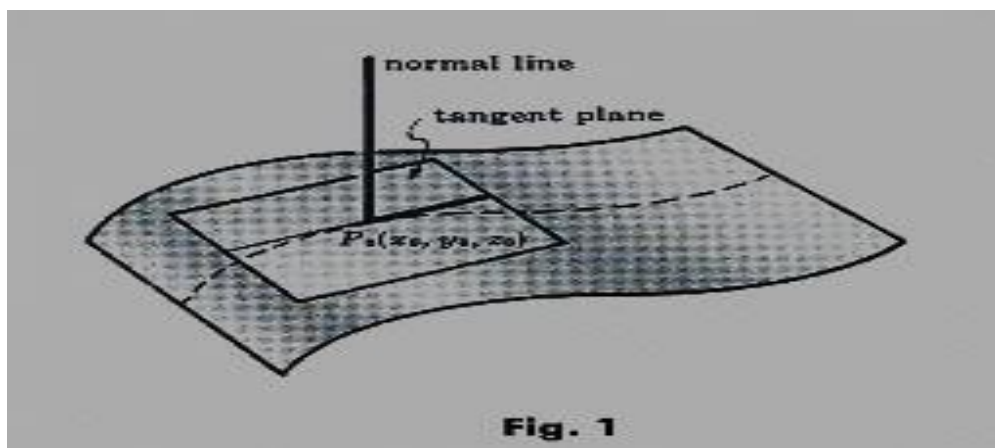


### 3.13 Tangent planes and normal lines



The **tangent plane** at the point  $P_0(x_0, y_0, z_0)$  on the level surface  $f(x, y, z) = C$  of a differentiable function  $f$  is the plane through  $P_0$  normal to  $\nabla f|_{P_0}$ .

The **normal line** of the surface at  $P_0$  is the line through  $P_0$  parallel to  $\nabla f|_{P_0}$ .

The tangent plane and normal line have the following equation:

**Tangent plane** to  $f(x, y, z) = C$  at  $P_0(x_0, y_0, z_0)$ :

$$f_x(p_0)(x - x_0) + f_y(p_0)(y - y_0) + f_z(p_0)(z - z_0) = 0$$

**Normal line** to  $f(x, y, z) = C$  at  $P_0(x_0, y_0, z_0)$ :

$$x = x_0 + f_x(p_0)t, \quad y = y_0 + f_y(p_0)t, \quad z = z_0 + f_z(p_0)t$$

**Example:** find the tangent plane and normal line of the surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \text{ at the point } P_o(1, 2, 4)$$

**Solution:** the tangent plane is:

$$f_x(p_o)(x - x_o) + f_y(p_o)(y - y_o) + f_z(p_o)(z - z_o) = 0$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z - 9) = 2x$$

$$f_x(P_o) = f_x(1, 2, 4) = (2)(1) = \boxed{2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + z - 9) = 2y$$

$$f_y(P_o) = f_y(1, 2, 4) = (2)(2) = \boxed{4}$$

$$f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2 + y^2 + z - 9) = 1$$

$$f_z(P_o) = f_z(1, 2, 4) = \boxed{1}$$

∴ The tangent plane is:

$$2(x-1) + 4(y-2) + (z-4) = 0$$

or

$$2x - 2 + 4y - 8 + z - 4 = 0$$

$$2x + 4y + z - 14 = 0$$

$$2x + 4y + z = 14$$

The normal line is:

$$x = x_o + f_x(p_o)t, \quad y = y_o + f_y(p_o)t, \quad z = z_o + f_z(p_o)t$$

$$\therefore x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t$$

*Example:* find the tangent plane and normal line of the surface  $f(x, y, z) = x^2 + y^2 + z^2 = 3$  at the point  $P_o(1,1,1)$

*Solution:* the tangent plane is:

$$f_x(p_o)(x - x_o) + f_y(p_o)(y - y_o) + f_z(p_o)(z - z_o) = 0$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2x$$

$$f_x(P_o) = f_x(1,1,1) = (2)(1) = \boxed{2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + z^2) = 2y$$

$$f_y(P_o) = f_y(1,1,1) = (2)(1) = \boxed{2}$$

$$f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2 + y^2 + z^2) = 2z$$

$$f_z(P_o) = f_z(1,1,1) = (2)(1) = \boxed{2}$$

∴ The tangent plane is:

$$2(x-1) + 2(y-1) + 2(z-1) = 0$$

or

$$2x - 2 + 2y - 2 + 2z - 2 = 0$$

$$2x + 2y + 2z - 6 = 0$$

$$2x + 2y + 2z = 6$$

$$2(x + y + z) = 6 \implies x + y + z = 3$$

The normal line is:

$$x = x_o + f_x(p_o)t, \quad y = y_o + f_y(p_o)t, \quad z = z_o + f_z(p_o)t$$

$$\therefore x = 1 + 2t, \quad y = 1 + 2t, \quad z = 1 + 2t$$

### Exercise:

*H.W:* find the tangent plane and normal line of the surface

$$f(x, y, z) = x^2 + 2xy - y^2 + z^2 = 7 \text{ at the point } P_o(1, -1, 3)$$