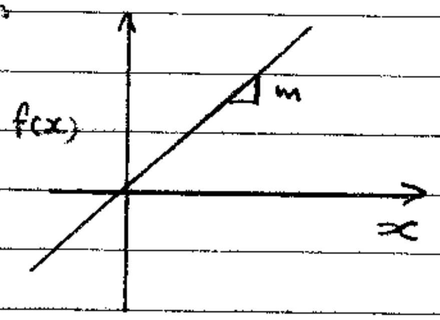
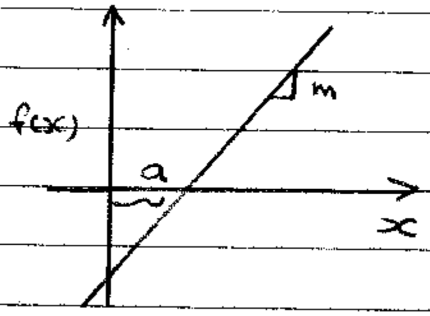
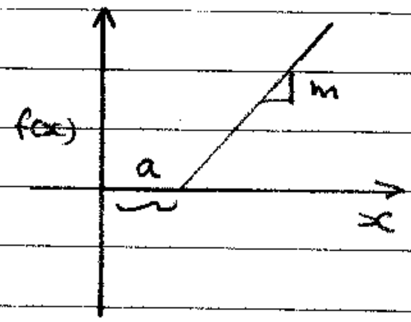
Uses of Unit Step Function :

Ex(1);

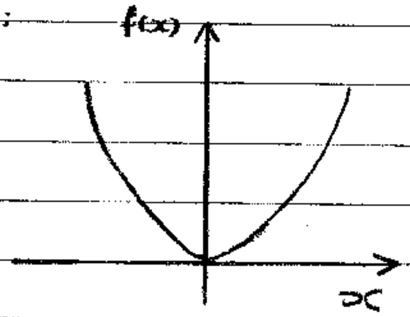


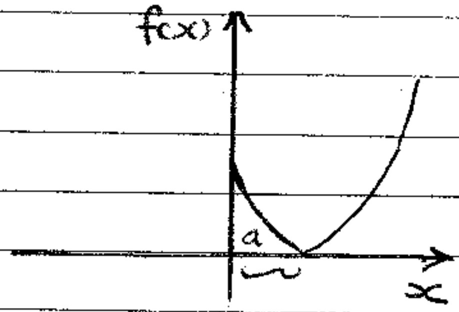


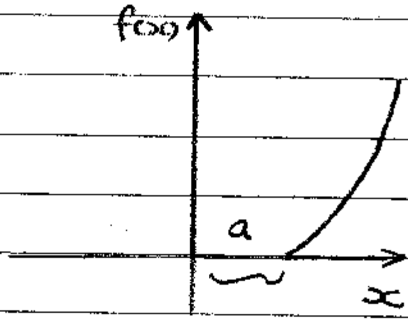


$f(x) = mx$ $\quad$ $f(x) = m(x-a)$ $\quad$ $f(x) = m(x-a) . S(x-a)$

Ex(2);

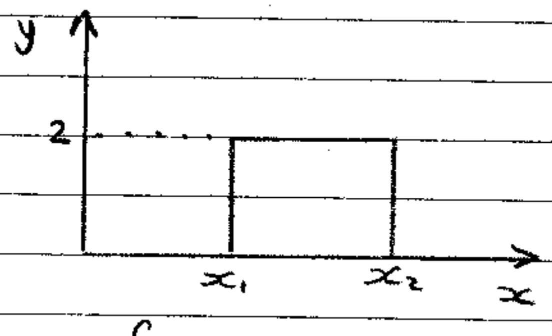






$f(x) = x^2$ $\quad$ $f(x) = (x-a)^2$ $\quad$ $f(x) = (x-a)^2 . S(x-a)$

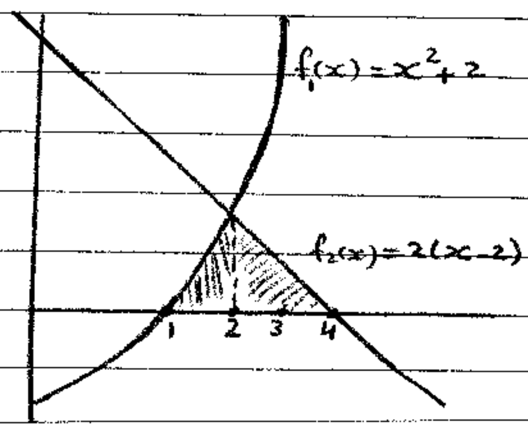
Ex(3) :



$$y = 2 \{ S(x - x_1) - S(x - x_2) \}$$

Ex(4) :

$$f(x) = \begin{cases} f_1(x) & 2 > x > 1 \\ f_2(x) & 4 > x > 2 \end{cases}$$



$$f(x) = (x^2+2)\left[S_1(x) - S_2(x) \right] + 2(x-2)\left[S_2(x) - S_4(x) \right]$$

Second Shifting Theorem:

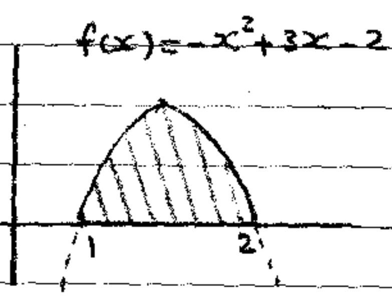
$$\mathcal{L}\, f(x).S(x-a) = e^{-as}\,\mathcal{L}\, f(x+a)$$

$$\mathcal{L}\, f(x-a)\, S(x-a) = e^{-as}\,\mathcal{L}\, f(x)$$

$$\mathcal{L}^{-1}(e^{-as}.\, f(x+a)) = f(x).\,S(x-a)$$

$$\mathcal{L}^{-1}(e^{-as}.\, f(x)) = f(x-a).\,S(x-a)$$

Example: Find the Laplace Transform for the following impulse function $f(x)$ given below.



$f(x) = (-x^2+3x-2)(S(x-1) - S(x-2))$

Used second shifting theorem

$$\mathcal{L}\, f(x) = e^{-s}\mathcal{L}\{-(x+1)^2+3(x+1)-2\}$$

$$- e^{-2s}\mathcal{L}\{-(x+2)^2+3(x+2)-2\}$$

$$= e^{-2s}\{\mathcal{L}(x^2+4x+4-3x-6+2)\} - e^{-s}\{\mathcal{L}(x^2+2x+1$$

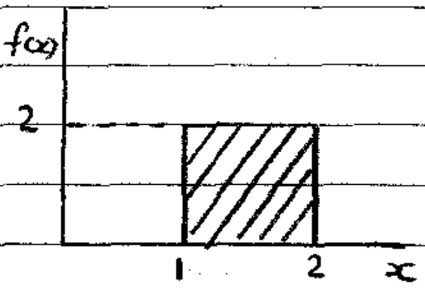
$$-3x-3+2)\}$$

$$= e^{-2s}\,\mathcal{L}(x^2+x) - e^{-s}\,\mathcal{L}(x^2-x)$$

$$= e^{-2s}\left[\frac{2!}{s^2} + \frac{1!}{s^2}\right] - e^{-s}\left[\frac{2!}{s^3} - \frac{1!}{s^2}\right]$$

Example: Solve the differential equation :

$$\frac{dy}{dx} + 3y + 2\int_0^x y\,dx = f(x)$$



where $y(0) = 0$ & $f(x)$ is given by the figure.

$$\frac{dy}{dx} + 3y + 2\int_0^x y\,dx = 2\left\{ S(x-1) - S(x-2) \right\}$$

Taking Laplace Transform

$$s\bar{y}(s) - y(0) + 3\bar{y}(s) + 2\,\frac{\bar{y}(s)}{s} = 2\left[\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right]$$

$$s\bar{y}(s) + 3\bar{y}(s) + 2\,\frac{\bar{y}(s)}{s} = \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s}$$

$$\bar{y}(s)\left(s + 3 + \frac{2}{s}\right) = \frac{2e^{-s} - 2e^{-2s}}{s}$$

$$\bar{y}(s)\left(\frac{s^2+3s+2}{s}\right) = \frac{2e^{-s} - 2e^{-2s}}{s}$$

$$\bar{y}(s)\left((s+1)(s+2)\right) = 2e^{-s} - 2e^{-2s}$$

$$\bar{y}(s) = 2e^{-s}\,\frac{1}{(s+1)(s+2)} - 2e^{-2s}\,\frac{1}{(s+1)(s+2)}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$1 = A(s+2) + B(s+1) \Rightarrow 1 = As + 2A + Bs + B$

$1 = (A+B)s + 2A + B \Rightarrow A + B = 0$ & $2A + B = 1$

$\therefore A = 1$ & $B = -1$

$$\bar{y}(s) = 2e^{-s}\,\frac{1}{s+1} - 2e^{-s}\,\frac{1}{s+2} - 2e^{-2s}\,\frac{1}{s+1} + 2e^{-2s}\,\frac{1}{s+2}$$

Used second shifting theorem

$$y(x) = 2e^{-(x-1)} \cdot S(x-1) - 2e^{-2(x-1)} \cdot S(x-1)$$

$$- 2e^{-(x-2)} \cdot S(x-2) + 2e^{-2(x-2)} \cdot S(x-2)$$

$$y(x) = 2e^{-x+1} \cdot S(x-1) - 2e^{-2x+2} \cdot S(x-1) - 2e^{-x+2} \cdot S(x-2)$$

$$+ 2e^{-2x+4} \cdot S(x-2)$$

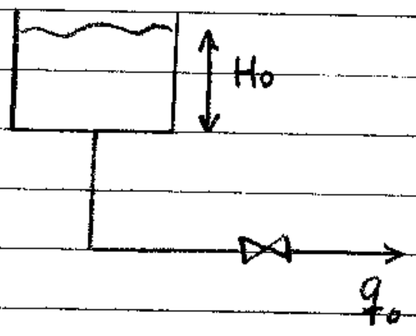
Formulation of Chemical Engineering Problems :

The mathematical model is an expression that represent a phenomenon or an operation. When deriving the model we make use of the basic theoretical principles and the validity of the model is, then, tested experimentally.

The main problems to be solved are :

1. Storage tanks.
2. Mixing tanks.
3. Chemical reaction vessels.
4. Heat transfer proplems.
5. Mass transfer problems.
6. Momentum transfer problems.
7. Process control systems.
8. Another problems.

Example: A vertical tank is filled with liquid to a height (H_o). The base of the tank is connected to a valve, if the valve is opend. Derive the equation which relate the variation of height with time, given that the flow through the valve is laminar.



Material balance on the tank

In = Out + Accumulation

$$0 = \rho q_o + \rho \frac{dV}{dt}$$

$$0 = \rho q_o + \rho A \frac{dH}{dt}$$

Laminar flow $\Rightarrow q_o \propto H \Rightarrow q_o = KH$, $K = \frac{m^3/hr}{m} = \frac{m^2}{hr}$

$$0 + KH + A\frac{dH}{dt} \Rightarrow \frac{A}{K}\frac{dH}{dt} + H = 0$$

$\tau \frac{dH}{dt} + H = 0$ Taking Laplace Transform

$$\tau[s\bar{H}(s) - H(0)] + \bar{H}(s) = 0$$

at $t=0$ $H = H_o$

$$\tau[s\bar{H}(s) - H_o] + \bar{H}(s) = 0$$

$$(\tau s + 1)\bar{H}(s) = \tau H_o \Rightarrow \bar{H}(s) = \frac{\tau H_o}{\tau s + 1} \Rightarrow \bar{H}(s) = \frac{\tau H_o}{\tau(s + \frac{1}{\tau})}$$

$\bar{H}(s) = \frac{H_o}{s + \frac{1}{\tau}}$ Taking Inverse Laplace Transform

$$H(t) = H_o e^{-\frac{1}{\tau}t} \Rightarrow H(t) = H_o e^{-\frac{t}{\tau}}$$

Another solution :

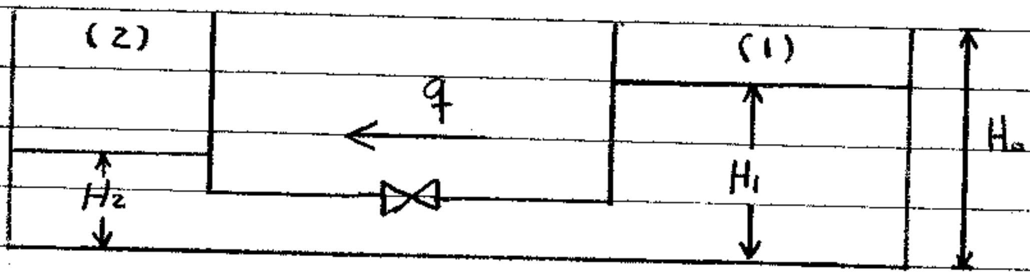
$$\tau \frac{dH}{dt} + H = 0 \Rightarrow \int \frac{dH}{H} = -\int \frac{dt}{\tau}$$

$$\ln H = -\frac{1}{\tau}t + \ln C \Rightarrow H = C e^{-t/\tau}$$

at $t=0$ $H = H_o \Rightarrow C = H_o$

$$H = H_o e^{-t/\tau} \quad \text{or} \quad H(t) = H_o e^{-t/\tau}$$

Example: Two tanks are connected as shown below. Tank 1 contain a liquid to height H_o and tank 2 is empty. The valve between the two tanks is opened. Find the relation which relate the height in tank 2 with time. Assuming that all resistance to flow was due to the valve and the flow is laminar.



Material balance on tank (1)

In = Out + Accumulation

$$0 = \rho q + \rho A_1 \frac{dH_1}{dt} \Rightarrow 0 = q + A_1 \frac{dH_1}{dt}$$

Material balance on tank (2)

In = Out + Accumulation

$$\rho q = 0 + \rho A_2 \frac{dH_2}{dt} \Rightarrow q = 0 + A_2 \frac{dH_2}{dt}$$

The flow is laminar $\Rightarrow q \propto H \Rightarrow q = KH$

or $q = K(H_1 - H_2)$

$$0 = K(H_1 - H_2) + A_1 \frac{dH_1}{dt}$$

$$K(H_1 - H_2) = 0 + A_2 \frac{dH_2}{dt}$$

By taking Laplace Transform

$$0 = K(\bar{H}_1(s) - \bar{H}_2(s)) + A_1(s\bar{H}_1(s) - H(0))$$
$$K(\bar{H}_1(s) - \bar{H}_2(s)) = 0 + A_2(s\bar{H}_2(s) - H(0))$$

At $t=0$ $H = H_0$ in tank (1)

At $t=0$ $H = 0$ in tank (2)

$$0 = K(\bar{H}_1(s) - \bar{H}_2(s)) + A_1(s\bar{H}_1(s) - H_0)$$
$$K(\bar{H}_1(s) - \bar{H}_2(s)) = 0 + A_2(s\bar{H}_2(s) - 0)$$
$$K\bar{H}_1(s) - K\bar{H}_2(s) = A_2 s\bar{H}_2(s)$$

$$\bar{H}_1(s) = \bar{H}_2(s) + \frac{A_2}{K} s\bar{H}_2(s) \Rightarrow \bar{H}_1(s) = \left(\frac{A_2}{K}s + 1\right)\bar{H}_2(s)$$

$$\bar{H}_2(s) = \bar{H}_1(s) + \frac{A_1}{K}(s\bar{H}_1(s) - H_0)$$

$$\bar{H}_2(s) = \left(\frac{A_1}{K}s + 1\right)\bar{H}_1(s) - \frac{A_1}{K}H_0$$

$$\bar{H}_2(s) = \left(\frac{A_1}{K}s + 1\right)\left(\frac{A_2}{K}s + 1\right)\bar{H}_2(s) - \frac{A_1}{K}H_0$$

let $\tau_1 = A_1/K$ & $\tau_2 = A_2/K$

$$\bar{H}_2(s) = (\tau_1 s + 1)(\tau_2 s + 1)\bar{H}_2(s) - \tau_1 H_0$$
$$(\tau_1 s + 1)(\tau_2 s + 1)\bar{H}_2(s) - \bar{H}_2(s) = \tau_1 H_0$$
$$[(\tau_1 s + 1)(\tau_2 s + 1) - 1]\bar{H}_2(s) = \tau_1 H_0$$

$$\bar{H}_2(s) = \frac{\tau_1 H_0}{(\tau_1 s + 1)(\tau_2 s + 1) - 1} \Rightarrow \bar{H}_2(s) = \frac{\tau_1 H_0}{\tau_1\tau_2 s^2 + \tau_1 s + \tau_2 s + 1 - 1}$$

$$\bar{H}_2(s) = \frac{\tau_1 H_0}{s(\tau_1\tau_2 s + \tau_1 + \tau_2)} \Rightarrow \bar{H}_2(s) = \frac{\tau_1 H_0}{s\,\tau_1\tau_2\left(s + \frac{\tau_1 + \tau_2}{\tau_1\tau_2}\right)}$$

let $K_1 = \dfrac{\tau_1 + \tau_2}{\tau_1 \tau_2}$

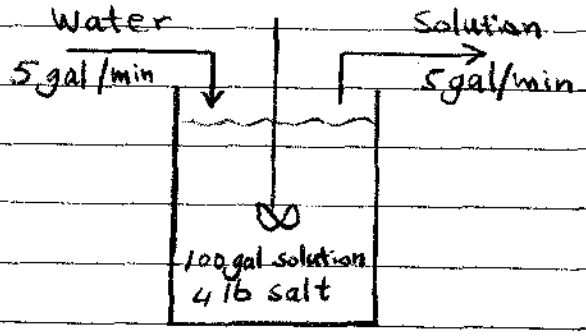
$$\bar{H}_2(s) = \frac{H_0}{\tau_2 s\,(s + K_1)} \Rightarrow$$

Taking Inverse Laplace Transform

$$H_2(t) = \frac{H_0}{\tau_2 K_1}\left(1 - e^{-K_1 t}\right)$$

$$H_2(t) = \frac{\tau_1 H_0}{\tau_1 + \tau_2}\left(1 - e^{-\left(\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}\right)t}\right)$$

Example: A tank holds 100 gal of water salt solution in which 4 lb of salt is dissolved. Water runs into the tank at the rate of 5 gal/min and salt solution overflows at the same rate. If the mixing in the tank is adequate to keep the concentration of salt in the tank uniform at all times, how much salt is in the tank at the end of 50 min?



Salt material balance

In = Out + Accumulation

$$0 = \frac{5x}{100} + \frac{dx}{dt}$$

Note: Out $= 5\,\dfrac{gal}{min} \cdot x\ lb \cdot \dfrac{1}{100\ gal} = \dfrac{5x}{100}\ \dfrac{lb}{min}$

where x is lb of salt in solution

$$\frac{dx}{dt} = -0.05x \Rightarrow \int \frac{dx}{x} = -0.05 \int dt$$

At $t = 0$ $\quad x = 4$

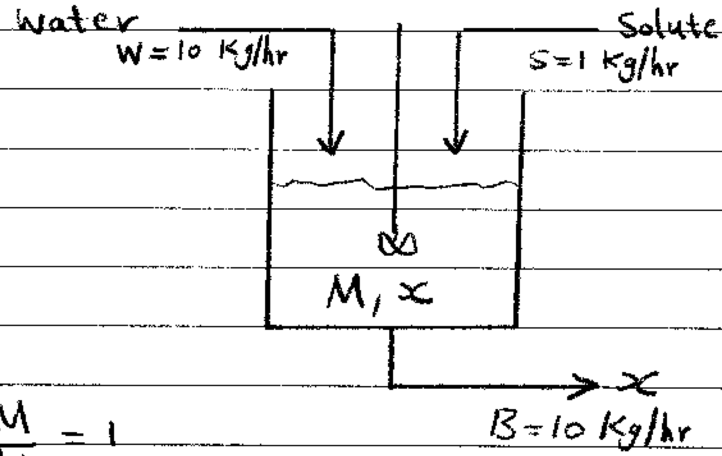
& $t = 50$ $\quad x = x$

$$\int_4^x \frac{dx}{x} = -0.05 \int_0^{50} dt \Rightarrow \ln x \Big]_4^x = -0.05\, t \Big]_0^{50}$$

$$\ln x - \ln 4 = -0.05\,(50 - 0) \Rightarrow \ln \frac{x}{4} = -0.05(50)$$

$$\ln \frac{x}{4} = -2.5 \Rightarrow x = 0.328 \text{ lb salt.}$$

Example: Water enter a mixing tank at a rate of $W = 10$ Kg/hr and solute added $S = 1$ Kg/hr, the exit stream is $B = 10$ Kg/hr. Initially, the tank containing $M_o = 100$ Kg water. Find the relation between change in concentration of solution in the tank with time?



Overall material balance

In = Out + Accumulation

$$W + S = B + \frac{dM}{dt}$$

$$10 + 1 = 10 + \frac{dM}{dt} \Rightarrow \frac{dM}{dt} = 1$$

$$\int dM = \int dt$$

At $t = 0$ $\quad M = M_o = 100$

At $t = t$ $\quad M = M$

$$\int_{100}^{M} dM = \int_{0}^{t} dt \Rightarrow M\Big]_{100}^{M} = t\Big]_{0}^{t} \Rightarrow M - 100 = t - 0$$

$$M = 100 + t$$

Solute material balance

In = Out + Accumulation

$$W(0) + S(1) = B(x) + \frac{d(Mx)}{dt}$$

$$1 = 10x + x\frac{dM}{dt} + M\frac{dx}{dt}$$

$$1 = 10x + x(1) + (100+t)\frac{dx}{dt}$$

$$1 = 11x + (100+t)\frac{dx}{dt} \Rightarrow (1 - 11x) = (100+t)\frac{dx}{dt}$$

$$\int \frac{dx}{(1-11x)} = \int \frac{dt}{(100+t)}$$

At $t = 0$ $x = 0$

At $t = t$ $x = x$

$$\int_{0}^{x} \frac{dx}{(1-11x)} = \int_{0}^{t} \frac{dt}{(100+t)}$$

$$-\frac{1}{11}\ln(1-11x)\Big]_{0}^{x} = \ln(100+t)\Big]_{0}^{t}$$

$$\frac{1}{11}\ln\left(\frac{1}{1-11x}\right)\Big]_{0}^{x} = \ln(100+t)\Big]_{0}^{t}$$

$$\ln\left(\frac{1}{1-11x}\right)^{1/11} = \ln\left(\frac{100+t}{100}\right) \Rightarrow \left(\frac{1}{1-11x}\right)^{1/11} = \frac{100+t}{100}$$