12.3. Problems :

1. find $L\left(e^{-t}\int_0^t t\cos t dt\right)$

Solution :

$$L\left(e^{-t}\int_0^t t\cos t dt\right) = \left[L\left(\int_0^t t\cos t dt\right)\right]_{s\to s+1}$$

$$= \left(\frac{1}{s}L(t\cos t)\right)_{s\to s+1}$$

$$= \left(\frac{1}{s}\left(\frac{-d}{ds}(L(\cos t))\right)\right)_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\frac{-d}{ds}\left(\frac{s}{s^2+1}\right)\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{(s^2+1)-s(2s)}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left(\frac{s^2-1}{s(s^2+1)^2}\right)_{s\to s+1}$$

$$= \left(\frac{(s+1)^2-1}{(s+1)((s+1)^2+1)^2}\right)_{s\to s+1}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

2. Find $L\left(e^{-t}\int_0^t \frac{\sin t}{t} dt\right)$

Solution :

$$L\left(e^{-t}\int_0^t \frac{\sin t}{t} dt\right) = \left[L\left(\int_0^t \frac{\sin t}{t} dt\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}L\left(\frac{\sin t}{t}\right)\right]_{s\to s+1}$$

Since $\lim_{t\to 0} \frac{\sin t}{t}$ exist

$$= \left[\frac{1}{s}\int_s^\infty L(\sin t) ds\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^\infty \frac{1}{s^2+1} ds\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\tan^{-1} s\right)_s^\infty\right]_{s\to s+1}$$

$$= \left[\frac{1}{s} (\tan^{-1} \infty - \tan^{-1}(s) \right]_{s \to s+1}$$

$$= \left[\frac{1}{s} \left(\frac{\pi}{2} - \tan^{-1}(s) \right) \right]_{s \to s+1}$$

$$= \left[\frac{1}{s} \cot^{-1} s \right]_{s \to s+1} \qquad = \frac{\cot^{-1}(s+1)}{s+1}$$

3. Find the Laplace Transform of $\int_0^t te^{-t} \sin t dt$

Solution :

$$L(te^{-t} \sin t dt) = (L(t \sin t))_{s \to s+1}$$

$$= \left(\frac{-d}{ds} L(\sin t) \right)_{s \to s+1}$$

$$= \left(\frac{-d}{ds} \left(\frac{1}{s^2+1} \right) \right)_{s \to s+1}$$

$$= -\left(\frac{(s^2+1)0 - 2s}{(s^2+1)^2} \right)_{s \to s+1}$$

$$= \left(\frac{2S}{(S^2+1)^2} \right)_{S \to S+1}$$

$$= \frac{2(S+1)}{((S+1)^2+1)^2}$$

$$= \frac{2(S+1)}{S^2+2S+2}$$

4. Find $L\left(\int_0^t \frac{e^{-t} \sin t}{t} dt \right)$

Solution :

$$L\left(\int_0^t \frac{e^{-t} \sin t}{t} dt \right) = \frac{1}{s} L\left(\frac{e^{-t} \sin t}{t} \right)$$

Since $\lim_{t \to 0} \frac{e^{-t} \sin t}{t}$ exist.

$$= \frac{1}{s} \left[\int_s^\infty L(e^{-t} \sin t) \right] ds$$

$$= \frac{1}{s} \left[\int_s^\infty L(\sin t) \right]_{s \to s+1} ds$$

$$= \frac{1}{s} \left[\int_s^\infty \left(\frac{1}{s^2+1} \right) \right]_{s \to s+1} ds$$

$$= \frac{1}{s} \left[\int_s^\infty \left(\frac{1}{(s+1)^2+1} \right) \right] ds$$

$$= \frac{1}{s}\left[\int_s^\infty \left(\frac{ds}{(s+1)^2+1}\right)\right]$$

$$= \frac{1}{s}\left(\tan^{-1}(s+1)\right)_s^\infty$$

$$= \frac{\cot^{-1}(s+1)}{s}$$

Problems :

1. Find $L\left(\int_0^t e^{2t}dt\right)$

Solution :

$$L\left(\int_0^t e^{2t}dt\right) = \frac{1}{s}L(e^{2t})$$

$$= \frac{1}{s}\cdot\frac{1}{s-2}$$

$$= \frac{1}{s(s-2)}$$

2. Find $L\left(\int_0^t \sin 3t dt\right)$

Solution :

$$L\left(\int_0^t \sin 3t dt\right) = \frac{1}{s}L(\sin 3t)$$

$$= \frac{1}{s}\cdot\frac{3}{s^2+9}$$

$$= \frac{3}{s(s^2+9)}$$

3. Find $L\left(\int_0^t e^{-2t}\cos 3t dt\right)$

Solution :

$$L\left(\int_0^t e^{-2t}\cos 3t dt\right) = \frac{1}{s}L(e^{-2t}\cos 3t)$$

$$= \frac{1}{s}L(\cos 3t)_{s\to s+2}$$ (Using first shifting theorem)

$$= \frac{1}{s}\left(\frac{s}{s^2+9}\right)_{s\to s+2}$$

$$= \frac{1}{s}\left(\frac{s+2}{(s+2)^2+9}\right)$$

4. Find $L\left(\int_0^t e^{-t}\sin h2tdt\right)$

Solution :

$$L\left(\int_0^t e^{-t}\sin h2tdt\right)=\frac{1}{s}L(e^{-t}\sin h2t)$$
$$=\frac{1}{s}L(\sin h2t)_{s\to s+1}$$
$$=\frac{1}{s}\left(\frac{2}{s^2-4}\right)_{s\to s+1}$$
$$=\frac{1}{s}\left(\frac{2}{(s+1)^2-4}\right)$$

5. Find $L\left(\int_0^t \sin 3t\cos 2tdt\right)$

Solution :

$$L\left(\int_0^t \sin 3t\cos 2tdt\right)=\frac{1}{s}L(\sin 3t\cos 2t)$$
$$=\frac{1}{2s}L(2\sin 3t\cos 2t)$$
$$=\frac{1}{2s}L(\sin 5t+\sin t)$$
$$=\frac{1}{2s}\left(\frac{5}{s^2+25}+\frac{1}{s^2+1}\right)$$

6. Find $L\left(e^{-3t}\int_0^t t\sin^2 t\right)dt$

Solution :

$$L\left(e^{-3t}\int_0^t t\sin^2 t\right)=L\left(\int_0^t t\sin^2 tdt\right)_{s\to s+3}$$
$$=\left[\frac{1}{s}L(t\sin^2 t)\right]_{s\to s+3}$$
$$=\left[\frac{-1}{s}\frac{d}{ds}L(\sin^2 t)\right]_{s\to s+3}$$
$$=\left[\frac{-1}{s}\frac{d}{ds}L\left(\frac{1-\cos 2t}{2}\right)\right]_{s\to s+3}$$
$$=\left[\frac{-1}{2s}\frac{d}{ds}L(1-\cos 2t)\right]_{s\to s+3}$$
$$=\left[\frac{-1}{2s}\frac{d}{ds}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\right]_{s\to s+3}$$
$$=\left[\frac{-1}{2s}\left(\frac{-1}{s^2}-\frac{(s^2+4)\cdot 1-s(2s)}{(s^2+4)^2}\right)\right]_{s\to s+3}$$

$$=\left[\frac{+1}{2s}\left(\frac{+1}{s^2}+\frac{4-s^2}{(s^2+4)^2}\right)\right]_{s\to s+3}$$

$$=\frac{1}{2(s+3)}\left(\frac{+1}{(s+3)^2}+\frac{4-(s+3)^2}{((s+3)^2+4)^2}\right)$$

$$=\frac{1}{2(s+3)^3}\left(\frac{4-(s+3)^2}{2(s+3)(s^2+6s+13)^2}\right)$$

7. Find $L\left(e^{4t}\left(\int_0^t \frac{\sin 3t\cos 2t}{t}dt\right)\right)$

Solution :

$$L\left(e^{4t}\left(\int_0^t \frac{\sin 3t\cos 2t}{t}dt\right)\right)$$

$$=L\left(\int_0^t \frac{\sin 3t\cos 2t}{t}dt\right)_{s\to s-4}$$

$$=\left[\frac{1}{s}L\left(\frac{\sin 3t\cos 2t}{t}\right)\right]_{s\to s-4}$$

$$=\left[\frac{1}{s}\int_s^\infty L(\sin 3t\cos 2t)dt\right]_{s\to s-4}$$

$$=\left[\frac{1}{2s}\int_s^\infty L(2\sin 3t\cos 2t)ds\right]_{s\to s-4}$$

$$=\left[\frac{1}{2s}\int_s^\infty L(\sin 5t+\sin t)ds\right]_{s\to s-4}$$

$$=\left[\frac{1}{2s}\int_s^\infty\left(\frac{5}{s^2+25}+\frac{1}{s^2+1}\right)ds\right]_{s\to s-4}$$

$$=\left[\frac{1}{2s}\left(5\cdot\frac{1}{5}\tan^{-1}\frac{s}{5}+\tan^{-1}s\right)\right]_{s\to s-4}$$

$$=\left[\frac{1}{2s}\left(\tan^{-1}\frac{s}{5}+\tan^{-1}s\right)_s^\infty\right]_{s\to s-4}$$

$$=\left[\frac{1}{2s}\left((\tan^{-1}\infty+\tan^{-1}\infty)-\left(\tan^{-1}\frac{s}{5}+\tan^{-1}s\right)\right)\right]_{s\to s-4}$$

$$=\left[\frac{1}{2s}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)-\tan^{-1}\frac{s}{5}-\tan^{-1}s\right]_{s\to s-4}$$

$$=\left[\frac{1}{2s}\left(\pi-\tan^{-1}\frac{s}{5}-\tan^{-1}s\right)\right]_{s\to s-4}$$

$$=\frac{1}{2(s-4)}\left(\pi-\tan^{-1}\frac{s-4}{5}-\tan^{-1}(s-4)\right)$$