UNIT III

APPLICATIONS OF LAPLACE TRANSFORM

1.1 INTRODUCTION

Laplace transform is a powerful integral transform used to switch a function from the time domain to the s - domain. It can greatly simplify the solution of problems involving differential equations. It is very useful in obtaining solution of linear differential equations both ordinary and partial, solution of system of simultaneous differential equations, solution of integral equations and in the evaluation of definite integral.

Ordinary and partial differential equations describe the way certain quantities vary with time such as the current in an electrical circuit, the oscillations of a vibrating membrane, or the flow of heat through an insulated conductor these equations are generally coupled with initial conditions that describe the state of the system at time t = 0. A very powerful technique for solving these problems is that of Laplace transform which transform the differential equation into an algebraic equation from which we get the solution.

Solutions of Differential Equations using Laplace Transform

The following results will be used in solving differential and integral equations using Laplace transforms.

Theorem :

If $f(t)$ is continuous in $t \geq o, f'(t)$ is piecewise continuous in every finite interval in the range $t \geq o$ and $f(t)$ and $f'(t)$ are of exponential order, then

$$L(f'(t)) = sL(f(t)) - f(0)$$

Proof :

The given conditions ensure the existence of the Laplace transforms of $f(t)$ and $f'(t)$.

By definition, $L(f'(t)) = \int_0^\infty e^{-st} f'(t)dt$

$$= \int_0^\infty e^{-st} d(f(t))$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt \text{ , on integration by parts}$$

$$= \lim_{t\to\infty}\left[e^{-st} f(t)\right] - f(0) + s.L(f(t))$$

$$= 0 - f(0) + sL(f(t)) \qquad [\because f(t) \text{ is of exponential order}]$$

$$= sL(f(t)) - f(0)$$

Corollary 1

In the above theorem if we replace $f(t)$ by $f'(t)$ we get,

$$L(f''(t)) = sL(f'(t)) - f'(0)$$

$$= s\left[sL(f(t)) - f(0)\right] - f'(0)$$

$$= s^2 L(f(t)) - sf(0) - f'(0)$$

Repeated application of the above theorem gives the following result:

$$L(f^n(t)) = s^n L(f(t)) - s^{n-1} f(0) - s^{n-2} f^1(0) - ..... - f^{n-1}(0)$$

Solved Problems :

1. Using Laplace transform, solve $y' - y = t, y(0) = 0$.

Solution :

Given $y' - y = t, y(0) = 0$

Taking Laplace transfrom on both sides,

$$L(y') - L(y) = L(t)$$

$$sL(y) - y(0) - L(y) = \frac{1}{s^2}$$

$$L(y)[S - 1] = \frac{1}{s^2}$$

$$L(y) = \frac{1}{s^2(s-1)}$$

$$\therefore y = L^{-1}\left[\frac{1}{s^2(s-1)}\right]$$

$$y = \int_0^t \int_0^t L^{-1}\left(\frac{1}{s-1}\right) dt\ dt$$

$$y = \int_0^t \int_0^t e^t\ dt\ dt$$

$$= \int_0^t \left[e^t\right]_0^t dt$$

$$= \int_0^t \left[e^t - 1\right]_0^t dt$$

$$= \left(e^t - t\right)_0^t$$

$$= e^t - t - 1$$

2. Solve $y'' - 4y' + 8y = e^{2t}, y(0) = 2$ and $y'(0) = -2$.

Solution :

Taking Laplace transforms on the sides of the equation, we get

$$L(y'') - 4L(y') + 8L(y) = L(e^{2t})$$

$$\left[s^2 L(y) - sy(0) - y^1(0)\right] - 4\left[sL(y) - y(0)\right] + 8L(y) = \frac{1}{s-2}$$

$$i.e., \left[s^2 - 4s + 8\right] L(y) = \frac{1}{s-2} + 2s - 10$$

$$L(y) = \frac{1}{(s-2)(s^2 - 4s + 8)} + \frac{2s - 10}{s^2 - 4s + 8}$$

$$= \frac{A}{s-2} + \frac{Bs + C}{s^2 - 4s + 8} + \frac{2s - 10}{s^2 - 4s + 8}$$

Solving we get $A=\frac{1}{4}, \quad B=\frac{-1}{4}, \quad C=\frac{1}{2}$

$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{-1}{4}s+\frac{1}{2}}{s^2-4s+8}+\frac{2s-10}{s^2-4s+8}$$
$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}s-\frac{19}{2}}{s^2-4s+8}$$
$$=\frac{\frac{1}{4}}{S-2}+\frac{\frac{7}{4}(S-2)-6}{(S-2)^2+4}$$
$$y=\frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right)+e^{2t}\left(\frac{\frac{7}{4}s-6}{s^2+4}\right)$$
$$=\frac{1}{4}e^{2t}+e^{2t}\left(\frac{7}{4}\cos 2t-3\sin 2t\right)$$
$$=\frac{1}{4}e^{2t}\left(1+7\cos 2t-12\sin 2t\right)$$

3. Use Laplace transform to solve $y'-y=e^t$ given that $y(0)=1$

Solution:

$$y'-y=e^t$$

Taking Laplace transform on both sides of the equation, we get $y'-y=t, y(0)=0$

$$\left[sL(y)-y(0)\right]-L(y)=\frac{1}{s-1}$$
$$L(y)[s-1]=\frac{1}{s-1}+1$$
$$L(y)=\frac{s}{(s-1)^2}$$
$$y=L^{-1}\left[\frac{s}{(s-1)^2}\right]$$
$$=L^{-1}\left[\frac{(s-1)+1}{(s-1)^2}\right]$$
$$=L^{-1}\left[\frac{1}{s-1}\right]+L^{-1}\left[\frac{1}{(s-1)^2}\right]$$
$$=e^t+te^t$$
$$=e^t(1+t)$$

4. Solve $\frac{d^2y}{dt^2}+9y=18t$ given that $y(0)=0=y\left(\frac{\pi}{2}\right)$

Solution :

$$y''+9y=18t \qquad \text{where} \quad y''=\frac{d^2y}{dt^2}$$

Taking Laplace transform on both sides of the equation, we get

$$L(y'')+9L(y)=18L(t)$$

$$\left[s^2L(y)-sy(0)-y'(0)\right]+9L(y)=\frac{18}{s^2}$$

$$L(y)\left[s^2+9\right]=\frac{18}{s^2}+y'(0)$$ [$\because y'(0)$ is not given we can take it to be a constant a]

$$=\frac{18}{s^2}+a$$

$$=\frac{as^2+18}{s^2}$$

$$L(y)=\frac{as^2+18}{s^2(s^2+9)}$$

$$=\frac{a}{s^2+9}+\frac{18}{s^2(s^2+9)}$$

$$y=L^{-1}\left(\frac{a}{s^2+9}\right)+L^{-1}\left(\frac{18}{s^2(s^2+9)}\right)$$

$$=L^{-1}\left(\frac{a}{s^2+9}\right)+L^{-1}\left(\frac{2}{s^2}-\frac{2}{(s^2+9)}\right)$$ (using partial fractions)

$$=\frac{a\sin 3t}{a}+2t-\frac{2\sin 3t}{3}$$

Now, using the conditions $y=0$ and $t=\frac{\pi}{2}$ we have

$$0 =\frac{a}{3}\sin\left(\frac{3\pi}{2}\right)+\pi-\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)$$

$$=-\frac{a}{3}+\pi+\frac{2}{3}$$

$$\frac{a}{3}=\frac{3\pi+2}{3}$$

Hence $a=3\pi+2$

$$\therefore y=\frac{(3\pi+2)\sin 3t}{3}+2t-\frac{2\sin 3t}{3}$$

$$=\pi\sin 3t+2t$$

Exercise :

1. Solve $y'' - 4y' + 8y = e^{2t}$, $y(0) = 2$ and $y'(0) = -2$

2. Solve $y'' + 4y = \sin wt$, $y(0) = 0$ and $y'(0) = 0$

3. Solve $y'' + y' - 2y = 3\cos 3t - 11\sin 3t$, $y(0) = 0$ and $y'(0) = 6$

4. Solve $(D^2 + 4D + 13)y = e^{-t}\sin t$, $y = 0$ and $Dy = 0$ at $t = 0$ where $D = \frac{d}{dt}$

5. Solve $(D^2 + 6D + 9)x = 6t^2 e^{-3t}$, $x = 0$ and $Dx = 0$ at $t = 0$.

6. Solve $x'' + 3x' + 2x = 2(t^2 + t + 1)$, $x(0) = 2$, $x'(0) = 0$.

7. Solve $y'' - 3y' - 4y = 2e^t$, $y(0) = y'(0) = 1$.

8. Solve $x'' + 9x = 18t$, $x(0) = 0$, $x\left(\frac{\pi}{2}\right) = 0$

9. $y'' + 4y' = \cos 2t$, $y(\pi) = 0$, $y'(\pi) = 0$.

10. $x'' - 2x' + x = t^2 e^{-3t}$, $x(0) = 2$, $x'(0) = 3$.

Answers :

1. $y = \frac{1}{4}e^{2t}(1 + 7\cos 2t - 12\sin 2t)$

2. $y = \frac{1}{8}(\sin 2t - 2t\cos 2t)$

3. $y = \sin 3t - e^{-2t} + e^t$

4. $y = \frac{1}{85}\left[e^{-t}\{-2\cos t + 9\sin t\}\right] + e^{-2t}\left\{2\cos 3t - \frac{7}{3}\sin 3t\right\}$

5. $x = \frac{1}{2}t^4 e^{-3t}$

6. $x = t^2 - 2t + 3 - e^{-2t}$

7. $y = \frac{1}{25}(13e^{-t} - 10te^{-t} + 12e^{4t})$

8. $x = 2t + \pi\sin 3t$

9. $y = \frac{1}{4}(t - \pi)\sin 2t$

10. $x = \left(\frac{t^4}{12} + t + 2\right)e^t$

Solution of Integral equations using Laplace transform

Theorem :

If $f(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and is of the exponential order, then

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

Proof :

Let $g(t) = \int_0^t f(t)dt$

$$\therefore \quad g^1(t) = f(t)$$

$$\therefore \quad L(g^1(t)) = sL(g(t)) - g(0)$$

$$i.e \quad L(f(t)) = sL\left(\int_0^t f(t)dt\right) - \int_0^0 f(t)dt$$

$$\therefore \quad L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

Corollary :

$$L\left[\int_0^t\int_0^t f(t)dtdt\right] = \frac{1}{s^2}L(f(t))$$

In generl

$$L\left[\int_0^t\int_0^t .....\int_0^t f(t)(dt)^n\right] = \frac{1}{s^n}L(f(t))$$

Problems :

1. Solve $y + \int_0^t ydt = t^2 + 2t$

Solution :

Given $y + \int_0^t ydt = t^2 + 2t$

Taking Laplace Transform on both sides

$$L(y) + L\left(\int_0^t ydt\right) = L(t^2) + L(2t)$$

$$L(y) + \frac{1}{s}L(y) = \frac{2}{s^3} + \frac{2}{s^2}$$

$$L(y)\left[1 + \frac{1}{s}\right] = 2\left[\frac{1+s}{s^3}\right]$$

$$L(y)\left[\frac{s+1}{s}\right]=2\left[\frac{s+1}{s^3}\right]$$

$$L(y)=2\left[\frac{s+1}{s^3}\right]\left[\frac{s}{s+1}\right]$$

$$=\frac{2}{s^2}$$

$$y=L^{-1}\left(\frac{2}{s^2}\right)=2t$$

2. Solve $\frac{dy}{dt}+2y+\int_0^t y\,dt=2\cos t, \quad y(0)=1$

Solution :

Given $\quad y'+2y+\int_0^t y\,dt=2\cos t$

Taking Laplace Transform on both sides

$$L(y')+2L(y)+L\left(\int_0^t y\,dt\right)=2L(\cos t)$$

$$sL(y)-y(0)+2L(y)+\frac{1}{s}L(y)=\frac{2s}{s^2+1}$$

$$L(y)\left[s+2+\frac{1}{s}\right]-1=\frac{2s}{s^2+1}$$

$$L(y)\left[\frac{s^2+2s+1}{s}\right]=\frac{2s}{s^2+1}+1$$

$$L(y)=\left[\frac{s^2+2s+1}{s^2+1}\right]\left[\frac{s}{s^2+2s+1}\right]$$

$$=\frac{s}{s^2+1}$$

$$y=L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t$$

3. Using Laplace Transform solve $y+\int_0^t y(t)\,dt=e^{-t}$

Solution :

Given $\quad y+\int_0^t y(t)\,dt=e^{-t}$

Taking Laplace transform on both sides,

$$L(y)+L\left(\int_0^t y(t)dt\right)=L(e^{-t})$$

$$L(y)+\frac{1}{s}L(y)=\frac{1}{s+1}$$

$$L(y)\left[1+\frac{1}{s}\right]=\frac{1}{s+1}$$

$$L(y)\left[\frac{s+1}{s}\right]=\frac{1}{s+1}$$

$$L(y)=\frac{s}{(s+1)^2}$$

$$y=L^{-1}\left(\frac{s}{(s+1)^2}\right)=L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$=L^{-1}\left(\frac{1}{s+1}\right)-e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$y=e^{-t}-e^{-t}t$$

$$y=e^{-t}(1-t)$$

4. Using Laplace transform, solve $x+\int_0^t x(t)dt=\cos t+\sin t$

Solution :

$$x+\int_0^t x(t)dt=\cos t+\sin t$$

Taking Laplace transform on both sides,

$$L(x)+L\left(\int_0^t x(t)dt\right)=L(\cos t+\sin t)$$

$$L(x)\left[1+\frac{1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)\left[\frac{s+1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)=\left(\frac{s+1}{s^2+1}\right)\left(\frac{s}{s+1}\right)$$

$$L(x)=\frac{s}{s^2+1}$$

$$\therefore x=L^{-1}\left(\frac{s}{s^2+1}\right)=\cos t$$

Solving Integral Equations using convolution

Theorem :

By the definition of convolution, we have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

and by convolution theorem, $L(f(t) * g(t)) = L(f(t))L(g(t))$

Problems :

1. Solve $y = 1 + 2\int_0^t e^{-2u} y(t-u)\, du$ ______(1)

Solution :

$\int_0^t e^{-2u} y(t-u)\, du$ is of the form $\int_0^t f(u)g(t-u)du$ where $f(t) = e^{-2t}, \quad g(t) = y(t)$

Taking Laplace Transform on both sides of (1),

$$L(y) = L(1) + 2L\left[\int_0^t e^{-2u} y(t-u)\, du\right]$$

$$= \frac{1}{s} + 2L\left[e^{-2t} * y(t)\right] \qquad \text{(Definition of convolution)}$$

$$= \frac{1}{s} + 2L(e^{-2t})L(y) \qquad \text{(Convolution theorem)}$$

$$= \frac{1}{s} + 2\left(\frac{1}{s+2}\right)L(y)$$

$$L(y) = \frac{1}{s} + \frac{2}{s+2}L(y)$$

$$L(y)\left[1 - \frac{2}{s+2}\right] = \frac{1}{s}$$

$$L(y)\left[\frac{s}{s+2}\right] = \frac{1}{s}$$

$$L(y) = \frac{s+2}{s^2} = \frac{1}{s} + \frac{2}{s^2}$$

$$y = L^{-1}\left(\frac{1}{s} + \frac{2}{s^2}\right)$$

$$y = 1 + 2t$$

2. Using Laplace transform solve $y = 1 + \int_0^t y(u)\sin(t-u)du$

Solution :

Given $y = 1 + \int_0^t y(u)\sin(t-u)du$

Taking Laplace transform on both sides,

$$L(y) = L(1) + L\left[\int_0^t y(u)\sin(t-u)du\right] \qquad \_\_\_\_(1)$$

Now the integral $\int_0^t y(u)\sin(t-u)du$ is of the form $\int_0^t f(u)\,g(t-u)du$ where $f(t) = y(t), \quad g(t) = \sin t$

$\therefore$ (1) becomes

$$L(y) = \frac{1}{s} + L(y(t) * \sin t)$$

$$L(y) = \frac{1}{s} + L(y)\cdot\frac{1}{s^2+1}$$

$$L(y)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s}$$

$$L(y)\left[\frac{s^2}{s^2+1}\right] = \frac{1}{s}$$

$$L(y) = \frac{s^2+1}{s^3}$$

$$= \frac{1}{s} + \frac{1}{s^3}$$

$$y = L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}L^{-1}\left(\frac{2}{s^3}\right)$$

$$y = 1 + \frac{1}{2}t^2$$

3. Using Laplace transform, solve $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$

Solution :

Given that $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du \qquad \_\_\_\_\_(1)$

Taking Laplace transform on both sides of (1),

$$L(f(t)) = L(\cos t) + L\left[\int_0^t e^{-u} f(t-u)du\right]$$

$$= \frac{s}{s^2+1} + L(e^{-t} * f(t))$$

$$= \frac{s}{s^2+1} + L(e^{-t})L(f(t))$$

$$= \frac{s}{s^2+1} + \frac{1}{s+1}L(f(t))$$

$$L(f(t))\left[1-\frac{1}{s+1}\right]=\frac{s}{s^2+1}$$
$$L(f(t))\left[\frac{s}{s+1}\right]=\frac{s}{s^2+1}$$
$$L(f(t))=\frac{s+1}{s^2+1}$$
$$f(t)=L^{-1}\left(\frac{s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s^2+1}\right)$$
$$f(t)=\cos t+\sin t$$

4. Solve the integral equation $y(t)=t^2+\int_0^t y(t)\sin(t-u)du$

Solution :

$$y(t)=t^2+\int_0^t y(t)\sin(t-u)du$$

Taking Laplace transform obn both sides,

$$L(y(t))=L(t^2)+L\left[\int_0^t y(t)\sin(t-u)du\right]$$

$$L(y)=\frac{2}{s^3}+L(y(t)*\sin t)$$
$$=\frac{2}{s^3}+L(y)L(\sin t)$$
$$=\frac{2}{s^3}+L(y)\left(\frac{1}{s^2+1}\right)$$
$$L(y)\left(1-\frac{1}{s^2+1}\right)=\frac{2}{s^3}$$
$$L(y)\left(\frac{s^2}{s^2+1}\right)=\frac{2}{s^3}$$
$$L(y)=\frac{2(s^2+1)}{s^5}=\frac{2}{s^3}+\frac{2}{s^5}$$
$$y=L^{-1}\left(\frac{2}{s^3}\right)+\frac{2}{4!}L^{-1}\left(\frac{4!}{s^5}\right)$$
$$y=t^2+\frac{1}{12}t^4$$

Exercise :

1. Solve $x' + 3x + 2\int_0^t x\,dt = t, \quad x(0) = 0$

2. Solve $y' + 4y + 5\int_0^t y\,dt = e^{-t} \quad y(0) = 0$

3. Solve $x' + 2x + \int_0^t x\,dt = \cos t, \quad x(0) = 1$

4. Solve $y' + 4y + 13\int_0^t y\,dt = 3e^{-2t}\sin 3t, \quad y(0) = 3$

5. Solve $x(t) = 4t - 3\int_0^t x(u)\sin(t-u)du$

6. Solve $y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)du$

7. Solve $\int_0^t y(u)y(t-u)du = 2y(t) + t - 2$

8. Solve $y(t) = t + \int_0^t \sin u\; y(t-u)du$

9. Solve $y = 1 + \int_0^t y(u)\sin(t-u)du$

10. Solve $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$

Answers :

1. $x = \frac{1}{2}(1 + e^{-2t}) - e^{-t}$

2. $y = \frac{-1}{2}e^{-t} + \frac{1}{2}e^{-t}(\cos t + 3\sin t)$

3. $x = \frac{1}{2}\left[(1-t)e^{-t} + \cos t\right]$

4. $y = e^{2t}\left[3\cos 3t - \frac{7}{3}\sin 3t + \frac{3}{2}t\sin 3t + t\cos 3t\right]$

5. $x = t + \frac{3}{2}\sin 2t$

6. $y(t) = e^{-t}\left(1-t\right)^2$

7. $y(t) = 1$

8. $y = t + \frac{t^3}{6}$

9. $y = 1 + \frac{t^2}{2}$

10. $f(t) = \cos t + \sin t$

Simulaneous differential equations

1. Using Laplace transform solve

$$\frac{dx}{dt} + y = \sin t$$
$$\frac{dy}{dt} + x = \cos t$$

given $x(0) = 2$ and $y(0) = 0$

Solution :

Applying Laplace transform to the given equations

We get, $L(x') + L(y) = L(\sin t)$

$L(y') + L(x) = L(\cos t)$

$$\therefore sL(x) - x(0) + L(y) = \frac{1}{s^2+1}$$

$$sL(y) - y(0) + L(x) = \frac{s}{s^2+1}$$

$$\therefore sL(x) + L(\text{y}) = \frac{1}{s^2+1} + 2$$

$$= \frac{2s^2+3}{s^2+1} \qquad \text{\_\_\_\_(1)}$$

Also $sL(y) + L(x) = \frac{s}{s^2+1}$ ____(2)

$$(1) \times s \Rightarrow s^2 l(x) + sL(y) = \frac{(2s^2+3)s}{s^2+1} \qquad \text{\_\_\_\_(3)}$$

$$(2) \Rightarrow \quad L(x) + sL(y) = \frac{s}{s^2+1} \qquad \text{\_\_\_\_(4)}$$

$$(3)\text{ - }(4)\ (s^2-1)L(x) = \frac{(2s^2+3)}{s^2+1} - \frac{s}{s^2+1}$$

$$= \frac{2s^3+2s}{s^2+1}$$

$$L(x) = \frac{2s}{s^2-1} \qquad \text{\_\_\_\_(5)}$$

Substituting (5) in (2), we get

$$sL(y) = \frac{s}{s^2+1} - \frac{2s}{s^2-1} = \frac{s(s^2-1) - 2s(s^2+1)}{(s^2+1)(s^2-1)}$$

$$= \frac{-s^3-3s}{(s^2+1)(s^2-1)}$$

$$= \frac{-s(s^2+3)}{-(s^2+1)(1-s^2)}$$

$$L(y) = \frac{(s^2+3)}{(s^2+1)(1-s^2)} \qquad \text{\_\_\_\_(6)}$$

From (5), $$x = L^{-1}\left(\frac{2s}{s^2-1}\right)$$
$$= 2\cosh t$$
$$y = L^{-1}\left(\frac{(s^2+3)}{(1-s^2)(s^2+1)}\right)$$

Consider $$\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{A}{1-s} + \frac{B}{1+s} + \frac{Cs+D}{s^2+1} \qquad \_\_\_\_(7)$$
$$s^2+3 = A(1+s)(s^2+1) + B(1-s)(s^2+1) + (Cs+D)(1-s)(1+s)$$

Put $s = 1, \quad 4 = A(2)(2)$
$$\Rightarrow 4 = 4A \Rightarrow A = 1$$

Put $s = -1, \quad 4 = B(2)(2)$
$$\Rightarrow B = 1$$

Put $s = 0, \quad 3 = A + B + D$
$$3 = 1 + 1 + D$$
$$\Rightarrow D = 1$$

Comparing the coefficient of S,

$$0 = A - B + C$$
$$\Rightarrow C = 0$$

Substituting the values of A, B, C, D in (7) we get

$$\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{1}{1-s} + \frac{1}{1+s} + \frac{1}{s^2+1}$$
$$\therefore y = L^{-1}\left(\frac{1}{1-s}\right) + L^{-1}\left(\frac{1}{1+s}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$
$$y = -e^{-t} + e^{-t} + \sin t$$

Hence the solution is $x = 2\cos ht$ and $y = -e^{-t} + e^{-t} + \sin t$

2. Solve $$\frac{dx}{dt} + ax = y$$
$$\frac{dy}{dt} + ay = x$$

given that $x = 0$ and $y = 1$ when $t = 0$

Solution :

Applying Laplace transform we get

$$L(x') + aL(x) = L(y)$$
$$L(y') + aL(y) = L(x)$$
$$\therefore sL(x) - x(0) + aL(x) = L(y)$$
$$sL(y) - y(0) + aL(y) = L(x)$$

Given that $x(0) = 0, \quad y(0) = 1$

$$\therefore sL(x) - x(0) + aL(x) = L(y)$$
$$sL(y) - y(0) + aL(y) = L(x)$$
$$\therefore sL(x) + aL(x) = L(y)$$
$$sL(y) - 1 + aL(y) = L(x)$$
$$\therefore (s+a)L(x) = L(y)$$
$$(s+a)L(x) - L(y) = 0 \qquad \_\_\_\_(1)$$
$$-L(x) + (s+a)L(y) = 1 \qquad \_\_\_\_(2)$$
$$(1) + (s+a) \times (2) \Rightarrow L(y)\left[(s+a)^2 - 1\right] = s + a$$
$$\therefore L(y) = \frac{s+a}{(s+a)^2 - 1}$$

Also by (1) $L(x) = \dfrac{1}{(s+a)^2 - 1}$

$$\therefore x = L^{-1}\left(\frac{1}{(s+a)^2 - 1}\right)$$
$$= e^{-at} L^{-1}\left(\frac{1}{s^2 - 1}\right)$$
$$= e^{-at} \sin ht$$
$$y = L^{-1}\left(\frac{s+a}{(s+a)^2 - 1}\right)$$
$$= e^{-at} L^{-1}\left(\frac{s}{s^2 - 1}\right)$$
$$= e^{-at} \cos ht$$

Exercise :

1. Solve the simultaneous equations

   $2x' - y' + 3x = 2t$ and $x' + 2y' - 2x - y = t^2 - t,\ x(0) = 1,\ y(0) = 1$

2. Solve the simultaneous equations

   $D^2x - Dy = \cos t$ and $Dx + D^2y = -\sin t; x = 1,\ Dx = 0,\ y = 0,\ Dy = 1$ at $t = 0$

3. Solve $x' - y = e^t$ and $y' + x = \sin t;\ x(0) = 1,\ y(0) = 0$.

4. Solve $x' - y = \sin t,\ y' - x = -\cos t;\ x = 2,$ and $y = 0$ at $t = 0$.

5. Solve $D^2x + y = -5\cos 2t,\ D^2y + x = 5\cos 2t,\ x = Dx = Dy = 1$ and $y = -1$ at $t = 0$.

Answers :

1. $x = -1 + \frac{9}{8}e^{-t} + \frac{7}{8}e^{\frac{3t}{5}}$

   $y = -\frac{9}{8}e^{-t} + \frac{49}{8}e^{\frac{3t}{5}} - t^2 - 3t - 4$

2. $x = 1 + t\sin t, \quad y = t\cos t$

3. $x = \frac{1}{2}(e^t + 2\sin t + \cos t - t\cos t)$

   $x = \frac{1}{2}(-e^t - \sin t + \cos t + t\sin t)$

4. $x = 2\cos ht, \quad y = 2\sin ht - \sin t$

5. $x = \sin t + \cos 2t, \quad y = \sin t - \cos 2t$