

$$C(x,t) = \frac{4(C_0 - C_i)}{\pi} \sum_{k=0}^{\infty} e^{-\frac{(2k+1)\pi\sqrt{D}}{L}t} \cdot \frac{1}{(2k+1)} \sin \frac{(2k+1)\pi}{L} x + C_i$$

Ex: Solve the partial differential equation,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

for the conditions,

i) $\theta(0,t) = 20$

ii) $\theta(20,t) = 20$

iii)
$$\theta(x,0) = \begin{cases} 120 & 0 \leq x \leq 15 \\ 30 & 15 \leq x \leq 20 \end{cases}$$

$$C(x,t) = \frac{4(C_0 - C_1)}{\pi} \sum_{k=0}^{\infty} e^{-\frac{(2k+1)\pi\sqrt{D}}{L}t} \cdot \frac{1}{(2k+1)} \sin \frac{(2k+1)\pi}{L} x + C_1$$

Ex: Solve the partial differential equation,

$$\frac{\partial \vartheta}{\partial t} = h^2 \frac{\partial^2 \vartheta}{\partial x^2}$$

for the conditions,

i) $\vartheta(0,t) = 20$

ii) $\vartheta(20,t) = 20$

iii)
$$\vartheta(x,0) = \begin{cases} 120 & 0 \leq x \leq 15 \\ 30 & 15 \leq x \leq 20 \end{cases}$$

$$\bar{\vartheta} = \vartheta - 20, \quad \bar{\vartheta}(x,t) = \vartheta(x,t) - 20$$

i) $\bar{\vartheta}(0,t) = 20 - 20 = 0$

ii) $\bar{\vartheta}(20,t) = 20 - 20 = 0$

iii)
$$\bar{\vartheta}(x,0) = \begin{cases} 120 - 20 = 100 & 0 \leq x \leq 15 \\ 30 - 20 = 10 & 15 \leq x \leq 20 \end{cases}$$

$$\frac{\partial \bar{\vartheta}}{\partial t} = h^2 \frac{\partial^2 \bar{\vartheta}}{\partial x^2}$$

The general solution

$$\bar{\vartheta}(x,t) = e^{-\beta^2 t} \left(A \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x \right)$$

B.C. 1 $x=0$ $\bar{\vartheta}=0$

$$0 = e^{-\beta^2 t} (A(1) + B(0)) \Rightarrow 0 = e^{-\beta^2 t} A$$

$$e^{-\beta^2 t} \neq 0 \Rightarrow A = 0$$

$$\bar{Q}(x, t) = e^{-\beta^2 t} \left(B \sin \frac{\beta}{h} x \right)$$

$$\text{B.C. 2 : } x = 20 \quad \bar{Q} = 0$$

$$0 = e^{-\beta^2 t} B \sin \frac{\beta}{h} (20)$$

$$e^{-\beta^2 t} \neq 0, B \neq 0 \Rightarrow \sin \frac{\beta}{h} (20) = 0$$

$$\frac{\beta}{h} 20 = n\pi \Rightarrow \beta = \frac{n\pi h}{20}$$

$$\bar{Q}(x, t) = B \sin \frac{n\pi}{20} x \cdot e^{-\left(\frac{n\pi h}{20}\right)^2 t}$$

More general

$$\bar{Q}(x, t) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{20} x \cdot e^{-\left(\frac{n\pi h}{20}\right)^2 t}$$

$$\text{I.C. : } \begin{array}{ll} t=0 & \bar{Q} = 100 \\ & \& \bar{Q} = 10 \end{array} \quad \begin{array}{l} 0 \leq x \leq 15 \\ 15 \leq x \leq 20 \end{array}$$

$$\bar{Q}(x, 0) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{20} x \cdot 1$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin nx \, dx$$

$$B_n = \frac{2}{20} \left[\int_0^{15} 100 \cdot \sin \frac{n\pi}{20} x \, dx + \int_{15}^{20} 10 \cdot \sin \frac{n\pi}{20} x \, dx \right]$$

$$B_n = \frac{2}{20} \left[\frac{20 \times 100}{n\pi} \left(-\cos \frac{n\pi}{20} x \right) \Big|_0^{15} + \frac{20 \times 10}{n\pi} \left(-\cos \frac{n\pi}{20} x \right) \Big|_{15}^{20} \right]$$

$$B_n = \frac{200}{n\pi} \left(1 - \cos \frac{3}{4} n\pi \right) + \frac{20}{n\pi} \left(\cos \frac{3}{4} n\pi - \cos n\pi \right)$$

$$\cos n\pi = (-1)^n$$

$$B_n = \frac{20}{n\pi} \left(10 - 10 \cos \frac{3}{4} n\pi + \cos \frac{3}{4} n\pi - \cos n\pi \right)$$

$$B_n = \frac{20}{n\pi} \left(10 - 9 \cos \frac{3}{4} n\pi - (-1)^n \right)$$

$$\bar{Q}(x,t) = \sum_{n=0}^{\infty} \frac{20}{n\pi} \left(10 - 9 \cos \frac{3}{4} n\pi - (-1)^n \right) \cdot \sin \frac{n\pi}{20} x \cdot e^{-\left(\frac{n\pi h}{20}\right)^2 t}$$

$$\bar{Q}(x,t) = Q(x,t) - 20 \Rightarrow Q(x,t) = \bar{Q}(x,t) + 20$$

$$\therefore Q(x,t) = 20 + \sum_{n=0}^{\infty} \frac{20}{n\pi} \left(10 - 9 \cos \frac{3n\pi}{4} - (-1)^n \right) \sin \frac{n\pi}{20} x \cdot e^{-\left(\frac{n\pi h}{20}\right)^2 t}$$

Ex: Solve the partial differential equation

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}, \quad h=1$$

For the following condition

$$\begin{aligned} \text{i)} \quad t=0 \quad \theta &= 0 & \theta(x, 0) &= 0 \\ \text{ii)} \quad x=0 \quad \theta &= 60 & \theta(0, t) &= 60 \\ \text{iii)} \quad x=100 \quad \frac{\partial \theta}{\partial x} &= 0 & \frac{\partial \theta}{\partial x}(100, t) &= 0 \end{aligned}$$

$$\bar{\theta}(x, t) = \theta(x, t) - 60$$

$$\frac{\partial \bar{\theta}}{\partial x} = \frac{\partial \theta}{\partial x} - 0 \Rightarrow \frac{\partial \bar{\theta}}{\partial x} = \frac{\partial \theta}{\partial x}$$

$$\begin{aligned} \text{i)} \quad t=0 \quad \bar{\theta} &= 0 - 60 = -60 & \bar{\theta}(x, 0) &= -60 \\ \text{ii)} \quad x=0 \quad \bar{\theta} &= 60 - 60 = 0 & \bar{\theta}(0, t) &= 0 \\ \text{iii)} \quad x=100 \quad \frac{\partial \bar{\theta}}{\partial x} &= 0 & \frac{\partial \bar{\theta}}{\partial x}(100, t) &= 0 \end{aligned}$$

$$\frac{\partial \bar{\theta}}{\partial t} = h^2 \frac{\partial^2 \bar{\theta}}{\partial x^2} \Rightarrow \frac{\partial \bar{\theta}}{\partial t} = \frac{\partial^2 \bar{\theta}}{\partial x^2}, \quad h=1$$

$$\bar{\theta}(x, t) = e^{-\beta^2 t} (A \cos \beta x + B \sin \beta x), \quad h=1$$

$$\text{B.C. 1} \quad x=0 \quad \bar{\theta} = 0$$

$$\bar{\theta}(0, t) = e^{-\beta^2 t} (A \cos(0) + B \sin(0)) = 0$$

$$\bar{\theta}(0, t) = e^{-\beta^2 t} \cdot A = 0$$

$$e^{-\beta^2 t} \neq 0 \Rightarrow A = 0$$

$$\bar{\Phi}(x,t) = e^{-\beta^2 t} (B \sin \beta x)$$

B.C. 2 $x=100$ $\frac{\partial \bar{\Phi}}{\partial x} = 0$

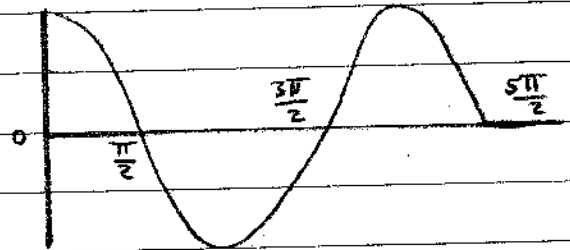
$$\frac{\partial \bar{\Phi}}{\partial x} = e^{-\beta^2 t} (B \cdot \beta \cdot \cos \beta x)$$

$$\frac{\partial \bar{\Phi}}{\partial x}(100,t) = e^{-\beta^2 t} \cdot B \cdot \beta \cdot \cos \beta(100) = 0$$

$$B \neq 0, e^{-\beta^2 t} \neq 0 \Rightarrow \cos \beta(100) = 0$$

$$100\beta = n \frac{\pi}{2}$$

$$\beta = \frac{n}{200} \pi$$



$$\bar{\Phi}(x,t) = e^{-\left(\frac{n}{200}\pi\right)^2 t} \cdot B \cdot \sin \frac{n}{200} \pi x$$

More general

$$\bar{\Phi}(x,t) = \sum_{n=0}^{\infty} e^{-\left(\frac{n}{200}\pi\right)^2 t} \cdot B_n \cdot \sin \frac{n}{200} \pi x$$

I.C. $t=0$ $\bar{\Phi} = -60$

$$\bar{\Phi}(x,0) = \sum_{n=0}^{\infty} 1 \cdot B_n \cdot \sin \frac{n}{200} \pi x = -60$$

This is a half Fourier series:

$$B_n = \frac{2}{L} \int_0^L f(x) \sin n\pi x \, dx$$

$$B_n = \frac{2}{100} \int_0^{100} (-60) \sin \frac{n}{200} \pi x \, dx$$

$$B_n = \frac{-2(60)}{100} \cdot \frac{200}{n\pi} \left[-\cos \frac{n\pi}{200} x \right]_0^{100}$$

$$B_n = \frac{240}{n\pi} \left[\cos \frac{n\pi}{2} - 1 \right]$$

put $n = 2K + 1$, $K = 0, 1, 2, 3, \dots$

$$B_K = \frac{240}{(2K+1)\pi} \left[\cos \frac{(2K+1)\pi}{2} - 1 \right]$$

$$\cos \frac{(2K+1)\pi}{2} = 0$$

$$\therefore B_K = \frac{-240}{(2K+1)\pi}$$

$$\bar{Q}(x,t) = \sum_{n=0}^{\infty} e^{-\frac{((2K+1)\pi)^2}{200} t} \cdot \frac{-240}{(2K+1)\pi} \sin \frac{(2K+1)\pi}{200} x$$

$$\bar{Q}(x,t) = Q(x,t) - 60 \Rightarrow Q(x,t) = \bar{Q}(x,t) + 60$$

$$Q(x,t) = 60 + \sum_{n=0}^{\infty} e^{-\frac{((2K+1)\pi)^2}{200} t} \cdot \frac{-240}{(2K+1)\pi} \sin \frac{(2K+1)\pi}{200} x$$

Ex: Solve the partial differential equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

for the following condition

i) $t=0$ $T = T_0$

ii) $x=0$ $T = T_0$

iii) $x=L$ $T = T_1$

Let $\bar{T} = T - T_s$ (T_s at steady state).

$$\frac{\partial^2 T}{\partial x^2} = 0 \Rightarrow \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial x} \right) = 0 \Rightarrow \frac{\partial T}{\partial x} = A$$

$$T = Ax + B$$

B.C. 1 $x=0$ $T=T_0$

$$T = Ax + B \Rightarrow T_0 = A(0) + B \Rightarrow B = T_0$$

B.C. 2 $x=L$ $T=T_1$

$$T = Ax + B \Rightarrow T_1 = AL + B \Rightarrow T_1 = AL + T_0$$

$$\therefore A = \frac{T_1 - T_0}{L}$$

$$T = Ax + B \Rightarrow T = \frac{T_1 - T_0}{L} x + T_0$$

$$\bar{T} = T - \left[\frac{T_1 - T_0}{L} x + T_0 \right]$$

$$\frac{\partial \bar{T}}{\partial t} = \alpha \frac{\partial^2 \bar{T}}{\partial x^2}$$

i) $t=0$ $\bar{T} = T_0 - \left[\frac{T_1 - T_0}{L} x + T_0 \right]$

$$\bar{T} = \frac{T_0 - T_1}{L} x$$

ii) $x=0$ $\bar{T} = T_0 - \left[\frac{T_1 - T_0}{L} (0) + T_0 \right] = 0$

iii) $x=L$ $\bar{T} = T_1 - \left[\frac{T_1 - T_0}{L} L + T_0 \right] = 0$

For $k = -ve$

$$\bar{T} = e^{-\beta^2 t} \left(A \cos \frac{\beta}{\sqrt{\alpha}} x + B \sin \frac{\beta}{\sqrt{\alpha}} x \right)$$

B.C. 1 $x=0$ $\bar{T}=0$

$$0 = e^{-\beta^2 t} (A \cos(0) + B \sin(0))$$

$$0 = e^{-\beta^2 t} \cdot A \Rightarrow e^{-\beta^2 t} \neq 0 \therefore A=0$$

$$\bar{T} = e^{-\beta^2 t} \cdot B \sin \frac{\beta}{\sqrt{\alpha}} x$$

B.C. 2 $x=L$ $\bar{T}=0$

$$0 = e^{-\beta^2 t} \cdot B \sin \frac{\beta}{\sqrt{\alpha}} L$$

$$e^{-\beta^2 t} \neq 0, B \neq 0 \Rightarrow \sin \frac{\beta}{\sqrt{\alpha}} L = 0$$

$$\frac{\beta}{\sqrt{\alpha}} L = n\pi, \quad n=0, 1, 2, 3, \dots$$

$$\beta = \frac{n\pi\sqrt{\alpha}}{L}$$

$$\bar{T} = e^{-\left(\frac{n\pi\sqrt{\alpha}}{L}\right)^2 t} \cdot B \cdot \sin \frac{n\pi}{L} x$$

More general

$$\bar{T} = \sum_{n=0}^{\infty} e^{-\left(\frac{n\pi\sqrt{\alpha}}{L}\right)^2 t} \cdot B_n \cdot \sin \frac{n\pi}{L} x$$

I.C. $t=0$ $\bar{T} = \frac{T_0 - T_1}{L} x$

$$\frac{T_0 - T_1}{L} x = \sum_{n=0}^{\infty} 1 \cdot B_n \cdot \sin \frac{n\pi}{L} x$$

$$B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin x \, dx$$

$$B_n = \frac{2}{L} \int_0^L \frac{T_0 - T_1}{L} x \cdot \sin \frac{n\pi}{L} x \, dx$$

$$B_n = \frac{2(T_0 - T_1)}{L^2} \int_0^L x \cdot \sin \frac{n\pi}{L} x \, dx$$

$$\int u \, dv = uv - \int v \, du$$

$$\int_0^L x \cdot \sin \frac{n\pi}{L} x \, dx = -x \frac{L}{n\pi} \cos \frac{n\pi}{L} x - \int \frac{L}{n\pi} \cos \frac{n\pi}{L} x \, dx$$

$$= \left[-\frac{xL}{n\pi} \cos \frac{n\pi}{L} x + \frac{L}{n\pi} \sin \frac{n\pi}{L} x \left(\frac{n\pi}{L} \right) \right]_0^L$$

$$= -\frac{xL}{n\pi} \cos \frac{n\pi}{L} x + \sin \frac{n\pi}{L} L$$

$$B_n = \frac{2(T_0 - T_1)}{L^2} \left[\sin \frac{n\pi}{L} L - \frac{xL}{n\pi} \cos \frac{n\pi}{L} x \right]$$

$$\bar{T} = \sum_{n=0}^{\infty} \frac{-2(T_0 - T_1)}{L^2} \cdot \frac{xL}{n\pi} \cos \frac{n\pi}{L} x \cdot e^{-\left(\frac{n\pi \sqrt{\alpha}}{L}\right)^2 t} \cdot \sin \frac{n\pi}{L} x$$

$$T = \sum_{n=0}^{\infty} \frac{-2(T_0 - T_1)}{L^2} \cdot \frac{xL}{n\pi} \cos \frac{n\pi}{L} x \cdot e^{-\left(\frac{n\pi \sqrt{\alpha}}{L}\right)^2 t} \cdot \sin \frac{n\pi}{L} x + \left[\frac{T_1 - T_0}{L} x + T_0 \right]$$

Ex: Find the general solution to the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{\partial u}{\partial t} \quad \text{cylinder equation}$$

$$u(r,t) = R(r) \cdot T(t)$$

$$\frac{\partial u}{\partial t} = RT'$$

$$\frac{\partial u}{\partial r} = TR' \quad \& \quad \frac{\partial^2 u}{\partial r^2} = TR''$$

$$TR'' + \frac{1}{r} TR' = RT' \quad \div RT$$

$$\frac{R''}{R} + \frac{R'}{rR} = \frac{T'}{T} = K$$

for $K < 0$ or $K = -\beta^2$

$$\frac{R'' + \frac{R'}{r}}{R} = -\beta^2 \Rightarrow R'' + \frac{R'}{r} + \beta^2 R = 0$$

$$r^2 R'' + rR' + r^2 \beta^2 R = 0$$

This is Bessel's equation of zero order, the solution for which is,

$$R = A J_0(r\beta) + B Y_0(r\beta)$$

$$\frac{T'}{T} = -\beta^2 \Rightarrow T' + \beta^2 T = 0 \Rightarrow T = C e^{-\beta^2 t}$$

The total solution is,

$$u(r,t) = e^{-\beta^2 t} (A J_0(r\beta) + B Y_0(r\beta))$$