Partial Differential Equations :

Partial differential equations are differential equations containing one dependent variable and two or more independent variables. There are many methods of solution for these equations.

1. Method of Direct Integration.
2. Separation of Variables (Forier Transforms).
3. Combination of Variables (Variation of Parameters).
4. Laplace Transforms.

Method of Direct Integration :

Ex: Solve the partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} = x e^y$$

For the boundary conditions,

$$u(0,y) = y^2$$
$$u(1,y) = \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = x e^y \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = x e^y$$

Integrating with respect to x

$$\frac{\partial u}{\partial x} = e^y \frac{x^2}{2} + F_1(y)$$

Integrating again,

$$u(x,y) = e^y \frac{x^3}{6} + x F_1(y) + F_2(y)$$

$F_1(y)$ and $F_2(y)$ are constants of integration with respect to x, but may be functions of y.

$$x = 0 \Rightarrow u = y^2$$

$$u(0, y) = F_2(y) = y^2$$

$$u(x, y) = \frac{x^3 e^y}{6} + x F_1(y) + y^2$$

$$x = 1 \Rightarrow u = \sin y$$

$$u(1, y) = \frac{e^y}{6} + F_1(y) + y^2 = \sin y$$

$$\therefore \; F_1(y) = \sin y - y^2 - \frac{e^y}{6}$$

$$u(x, y) = \frac{x^3 e^y}{6} + x\left(\sin y - y^2 - \frac{e^y}{6}\right) + y^2$$

Separation of Variables :

The solution starts by assuming the solution is a product of functions of the independent variables.

Ex: Find the general solutions for the equation :

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

Assume : $\quad C(x, t) = X(x) \cdot T(t)$

$$\frac{\partial c}{\partial t} = X \cdot T'$$

$$\frac{\partial c}{\partial x} = T \cdot X' \quad \& \quad \frac{\partial^2 c}{\partial x^2} = T \cdot X''$$

$$X T' = D X'' T \Rightarrow \frac{T'}{T} = D \frac{X''}{X} = \text{constant}$$

$$\frac{T'}{T} = D \frac{X''}{X} = K$$

There are three cases for K

Case (1): $K > 0 \Rightarrow K = \alpha^2$

$$\frac{T'}{T} = \alpha^2 \Rightarrow \ln T = \alpha^2 t + \ln \bar{C} \Rightarrow \ln T - \ln \bar{C} = \alpha^2 t$$

$$\ln \frac{T}{\bar{C}} = \alpha^2 t \Rightarrow \frac{T}{\bar{C}} = e^{\alpha^2 t} \Rightarrow T = \bar{C} e^{\alpha^2 \cdot t}$$

$$D \frac{X''}{X} = \alpha^2 \Rightarrow X'' = \frac{\alpha^2}{D} X \Rightarrow X'' - \frac{\alpha^2}{D} X = 0$$

$$m^2 - \frac{\alpha^2}{D} = 0 \Rightarrow m = \mp \frac{\alpha}{\sqrt{D}}$$

$$X = \bar{A} e^{\frac{\alpha}{\sqrt{D}} X} + \bar{B} e^{-\frac{\alpha}{\sqrt{D}} X}$$

$$C(x,t) = \bar{C} e^{\alpha^2 \cdot t} \left(\bar{A} e^{\frac{\alpha}{\sqrt{D}} x} + \bar{B} e^{-\frac{\alpha}{\sqrt{D}} x} \right)$$

$$C(x,t) = e^{\alpha^2 \cdot t} \left(A e^{\frac{\alpha}{\sqrt{D}} x} + B e^{-\frac{\alpha}{\sqrt{D}} x} \right)$$

Case (2): $K = 0$

$$\frac{T'}{T} = 0 \Rightarrow T' = 0 \Rightarrow T = \bar{A}$$

$$D \frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow X' = \bar{B} \Rightarrow X = \bar{B} X + \bar{C}$$

$$C(x,t) = \bar{A} (\bar{B} X + \bar{C}) \Rightarrow C(x,t) = A X + B$$

case (3): $K < 0 \Rightarrow K = -\beta^2$

$$\frac{T'}{T} = -\beta^2 \Rightarrow \ln T = -\beta^2 t + \ln \bar{C}$$

$$\ln \frac{T}{\bar{C}} = -\beta^2 . t \Rightarrow T = \bar{C}\, e^{-\beta^2 . t}$$

$$D\frac{X''}{X} = -\beta^2 \Rightarrow X'' = \frac{-\beta^2}{D} X \Rightarrow X'' + \frac{\beta^2}{D} X = 0$$

$$m^2 + \frac{\beta^2}{D} = 0 \Rightarrow m^2 = -\frac{\beta^2}{D} \Rightarrow m = \mp i \frac{\beta}{\sqrt{D}}$$

$$X = \bar{A} \cos \frac{\beta}{\sqrt{D}} x + \bar{B} \sin \frac{\beta}{\sqrt{D}} x$$

$$C(x,t) = \bar{C}\, e^{-\beta^2 . t} \left(\bar{A} \cos \frac{\beta}{\sqrt{D}} x + \bar{B} \sin \frac{\beta}{\sqrt{D}} x \right)$$

$$C(x,t) = e^{-\beta^2 . t} \left(A \cos \frac{\beta}{\sqrt{D}} x + B \sin \frac{\beta}{\sqrt{D}} x \right)$$

Ex: Solve the partial differential equation,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

for the following conditions,

i) $t = 0 \quad \theta = 100\ ^\circ C$

ii) $x = 0 \quad \theta = 0\ ^\circ C$

iii) $x = 1 \quad \theta = 0\ ^\circ C$

Assume: $\theta(x,t) = X(x) . T(t)$

$$\frac{\partial \theta}{\partial t} = X . T'$$

$$\frac{\partial \theta}{\partial x} = T \cdot X' \quad \& \quad \frac{\partial^2 \theta}{\partial x^2} = T \cdot X''$$

$$XT' = h^2 X'' T \Rightarrow \frac{T'}{T} = h^2 \frac{X''}{X} = \text{constant}$$

$$\frac{T'}{T} = h^2 \frac{X''}{X} = k$$

Case (3) : $K < 0 \Rightarrow K = -\beta^2$

$$\frac{T'}{T} = -\beta^2 \Rightarrow T = \bar{C}\, e^{-\beta^2 \cdot t}$$

$$h^2 \frac{X''}{X} = -\beta^2 \Rightarrow X = \bar{A} \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x$$

$$\theta(x,t) = e^{-\beta^2 \cdot t} \left(A \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x \right)$$

To find the constants $A, B, \& \beta$:

B.C.1 $\quad x = 0 \quad \theta = 0$

$$0 = e^{-\beta^2 \cdot t} \left(A \cos \frac{\beta}{h}(0) + B \sin \frac{\beta}{h}(0) \right)$$

$$0 = e^{-\beta^2 \cdot t} \left(A(1) + B(0) \right) \Rightarrow 0 = A e^{-\beta^2 \cdot t}$$

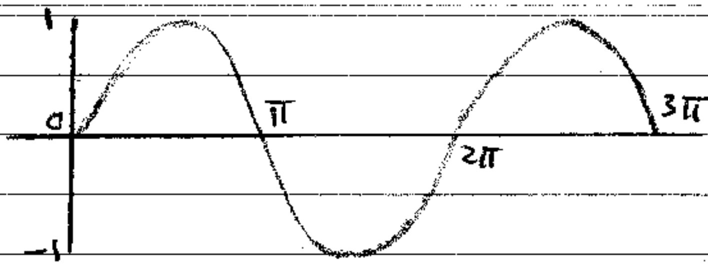
$$e^{-\beta^2 \cdot t} \neq 0 \Rightarrow A = 0$$

$$\theta = e^{-\beta^2 \cdot t} B \sin \frac{\beta}{h} x$$

B.C.2 $\quad x = 1 \quad \theta = 0$

$$0 = e^{-\beta^2 \cdot t} B \sin \frac{\beta}{h} (1)$$

$$e^{-\beta^2 \cdot t} \neq 0 \quad , \quad \beta \neq 0$$



$$\therefore \sin \frac{\beta}{h} = 0$$

$$\frac{\beta}{h} = n\pi \Rightarrow \beta = n\pi h \quad , \quad n = 0, 1, 2, 3, \ldots$$

$$\theta = e^{-(n\pi h)^2 \cdot t} \cdot B \sin n\pi x$$

More general

$$\theta = \sum_{n=0}^{\infty} e^{-(n\pi h)^2 \cdot t} \cdot B_n \cdot \sin n\pi x$$

I.C. $\quad t = 0 \quad \theta = 100$

$$100 = \sum_{n=0}^{\infty} 1 \cdot B_n \cdot \sin n\pi x$$

This is a half fourier series :

$$B_n = \frac{2}{L} \int_0^L f(x) \sin nx \, dx$$

$$B_n = \frac{2}{1} \int_0^1 100 \sin n\pi x \, dx$$

$$B_n = 200 \left[\frac{-\cos n\pi x}{n\pi} \right]_0^1 \Rightarrow B_n = \frac{-200}{n\pi} (\cos n\pi - 1)$$

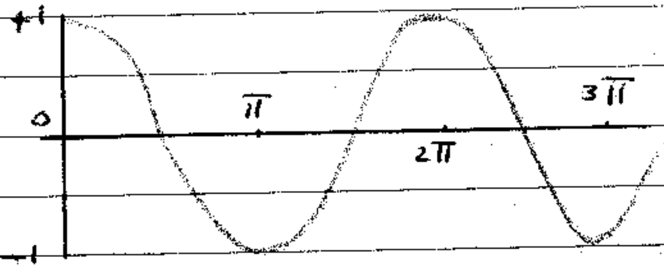
$$B_n = \frac{-200}{n\pi} \left((-1)^n - 1 \right)$$

put $n = 2K + 1$

$$B_n = \frac{-200}{(2K+1)\pi}\left((-1)^{2K+1} - 1\right)$$

$K = 0, 1, 2, 3, \ldots$

$$B_n = \frac{400}{(2K+1)\pi}$$



$$\theta = \sum_{K=0}^{\infty} e^{-((2K+1)\pi h)^2 \cdot t} \cdot \frac{400}{(2K+1)\pi} \cdot \sin(2K+1)\pi x$$

$$\theta = \frac{400}{\pi} \sum_{K=0}^{\infty} \frac{1}{2K+1} e^{-((2K+1)\pi h)^2 \cdot t} \cdot \sin(2K+1)\pi x$$

Ex: Solve the partial differential equation,

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}$$

for the following conditions:

i) $C(x, 0) = C_o$
ii) $C(0, t) = C_i$
iii) $C(L, t) = C_i$

Case (3): $K < 0 \Rightarrow K = -\beta^2$

$$C(x, t) = e^{-\beta^2 \cdot t}\left(A \cos \frac{\beta}{\sqrt{D}} x + B \sin \frac{\beta}{\sqrt{D}} x\right)$$

Let $\bar{C} = C - C_i$

$$\frac{\partial \bar{C}}{\partial t} = D \frac{\partial^2 \bar{C}}{\partial x^2}$$

i) $\bar{C}(x,0) = C_0 - C_i$

ii) $\bar{C}(0,t) = C_i - C_i = 0$

iii) $\bar{C}(L,t) = C_i - C_i = 0$

$$\bar{C}(x,t) = e^{-\beta^2 \cdot t}\left(A \cos \frac{\beta}{\sqrt{D}} x + B \sin \frac{\beta}{\sqrt{D}} x\right)$$

B.C. 1 : $x = 0$ $\quad \bar{C} = 0$

$$0 = e^{-\beta^2 \cdot t}\left(A(1) + B(0)\right) \Rightarrow 0 = A\, e^{-\beta^2 \cdot t}$$

$$e^{-\beta^2 \cdot t} \neq 0 \Rightarrow A = 0$$

$$\bar{C}(x,t) = e^{-\beta^2 \cdot t} \cdot B \sin \frac{\beta}{\sqrt{D}} x$$

B.C. 2 : $x = L$ $\quad \bar{C} = 0$

$$0 = e^{-\beta^2 \cdot t} \cdot B \sin \frac{\beta}{\sqrt{D}} L$$

$$e^{-\beta^2 \cdot t} \neq 0 , \quad B \neq 0 \Rightarrow \sin \frac{\beta}{\sqrt{D}} L = 0$$

$$\frac{\beta}{\sqrt{D}} L = n\pi \Rightarrow \beta = \frac{n \cdot \pi \cdot \sqrt{D}}{L}$$

$$\bar{C}(x,t) = e^{-\left(\frac{n\pi\sqrt{D}}{L}\right)^2 \cdot t}\left(B \sin \frac{n\pi}{L} x\right)$$

More general solution

$$\bar{C}(x,t) = \sum_{n=0}^{\infty} e^{-\left(\frac{n\pi\sqrt{D}}{L}\right)^2 \cdot t} \cdot B_n \cdot \sin\left(\frac{n\pi}{L}\right) x$$

I.C. : $t=0$ $\quad \bar{C} = C_o - C_i$

$$\bar{C}(x,0) = \sum_{n=0}^{\infty} 1 \cdot B_n \sin\left(\frac{n\pi}{L}\right)x = C_o - C_i$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin nx \, dx$$

$$B_n = \frac{2}{L}\int_0^L (C_o - C_i)\sin\left(\frac{n\pi}{L}\right)x \, dx$$

$$B_n = \frac{2(C_o - C_i)}{L}\left[-\frac{L}{n\pi}\left(\cos\frac{n\pi}{L}x\right)\right]_0^L$$

$$B_n = \frac{-2(C_o - C_i)}{n\pi}\left(\cos n\pi - 1\right)$$

$$B_n = \frac{-2(C_o - C_i)}{n\pi}\left((-1)^n - 1\right)$$

put $n = 2K+1$

$$B_n = \frac{-2(C_o - C_i)}{(2K+1)\pi}\left((-1)^{2K+1} - 1\right)$$

$K = 0, 1, 2, 3, \ldots$

$$B_n = \frac{-2(C_o - C_i)}{(2K+1)\pi}(-2) \Rightarrow B_n = \frac{4(C_o - C_i)}{(2K+1)\pi}$$

$$\bar{C}(x,t) = \sum_{K=0}^{\infty} e^{-\left(\frac{(2K+1)\pi\sqrt{D}}{L}\right)^2 \cdot t} \cdot \frac{4(C_o - C_i)}{(2K+1)\pi}\sin\frac{(2K+1)\pi}{L}x$$

$\because \bar{C} = C - C_i \Rightarrow C = \bar{C} + C_i$

$$C(x,t) = \frac{4(C_0 - C_i)}{\pi} \sum_{K=0}^{\infty} e^{-\left(\frac{(2K+1)\pi\sqrt{D}}{L}\right)^2 \cdot t} \cdot \frac{1}{(2K+1)} \sin \frac{(2K+1)\pi}{L} x + C_i$$

Ex: Solve the partial differential equation,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

for the conditions,

i) $\theta(0,t) = 20$

ii) $\theta(20,t) = 20$

iii)
$$\theta(x,0) = \begin{cases} 120 & 0 \leqslant x \leqslant 15 \\ 30 & 15 \leqslant x \leqslant 20 \end{cases}$$