Ex: Find the general solution to the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t} \qquad \text{Cylinder equation}$$

$$u(r,t) = R(r)\,.\,T(t)$$

$$\frac{\partial u}{\partial t} = RT'$$

$$\frac{\partial u}{\partial r} = TR' \qquad \& \qquad \frac{\partial^2 u}{\partial r^2} = TR''$$

$$TR'' + \frac{1}{r}TR' = RT' \qquad \div RT$$

$$\frac{R''}{R} + \frac{R'}{rR} = \frac{T'}{T} = K$$

for $K < 0$ or $K = -\beta^2$

$$\frac{R'' + \frac{R'}{r}}{R} = -\beta^2 \Rightarrow R'' + \frac{R'}{r} + \beta^2 R = 0$$

$$r^2 R'' + rR' + r^2\beta^2 R = 0$$

This is Bessel's equation of zero order, the solution for which is,

$$R = A\,J_0(r\beta) + B\,Y_0(r\beta)$$

$$\frac{T'}{T} = -\beta^2 \Rightarrow T' + \beta^2 T = 0 \Rightarrow T = C\,e^{-\beta^2 t}$$

The total solution is,

$$u(r,t) = e^{-\beta^2 t}\left(AJ_0(r\beta) + BY_0(r\beta)\right)$$

Ex: Solve the partial differential equation

$$\frac{\partial^2 c}{\partial r^2} + \frac{2}{r}\frac{\partial c}{\partial r} = \frac{1}{D}\frac{\partial c}{\partial t} \qquad \text{sphere equation}$$

For the conditions,

i) $t=0 \qquad c=10$

ii) $r=0 \qquad \frac{\partial c}{\partial r}=0$

iii) $r=1 \qquad c=0$

Let $c = \frac{\bar{c}}{r}$

$$\frac{\partial c}{\partial r} = \frac{\partial \bar{c}}{\partial r}\cdot\frac{1}{r} - \frac{1}{r^2}\bar{c}$$

$$\frac{\partial^2 c}{\partial r^2} = \frac{1}{r}\frac{\partial^2 \bar{c}}{\partial r^2} - \frac{1}{r^2}\frac{\partial \bar{c}}{\partial r} + 2\frac{\bar{c}}{r^3} - \frac{1}{r^2}\frac{\partial \bar{c}}{\partial r}$$

$$\frac{\partial^2 c}{\partial r^2} = \frac{1}{r}\frac{\partial^2 \bar{c}}{\partial r^2} - \frac{2}{r^2}\frac{\partial \bar{c}}{\partial r} + \frac{2\bar{c}}{r^3}$$

$$\frac{\partial c}{\partial t} = \frac{1}{r}\frac{\partial \bar{c}}{\partial t}$$

Sub. in equation

$$\frac{1}{r}\frac{\partial^2 \bar{c}}{\partial r^2} - \frac{2}{r^2}\frac{\partial \bar{c}}{\partial r} + \frac{2\bar{c}}{r^3} + \frac{2}{r}\left(\frac{\partial \bar{c}}{\partial r}\cdot\frac{1}{r} - \frac{1}{r^2}\bar{c}\right) = \frac{1}{D}\frac{1}{r}\frac{\partial \bar{c}}{\partial t}$$

$$\frac{\cancel{1}}{\cancel{r}}\frac{\partial^2 \bar{c}}{\partial r^2} - \cancel{\frac{2}{r^2}\frac{\partial \bar{c}}{\partial r}} + \cancel{\frac{2\bar{c}}{r^3}} + \cancel{\frac{2}{r^2}\frac{\partial \bar{c}}{\partial r}} - \frac{2\bar{c}}{r^3} = \frac{1}{D}\frac{\cancel{1}}{\cancel{r}}\frac{\partial \bar{c}}{\partial t}$$

$$\frac{\partial^2 \bar{c}}{\partial r^2} = \frac{1}{D}\frac{\partial \bar{c}}{\partial t} \quad \text{or} \quad \frac{\partial \bar{c}}{\partial t} = D\frac{\partial^2 \bar{c}}{\partial r^2}$$

for $K<0$

$$\bar{c}(r,t) = e^{-\beta^2 t}\left(A\cos\frac{\beta}{\sqrt{D}}r + B\sin\frac{\beta}{\sqrt{D}}r\right)$$

B.C. 1 $\quad r=0 \quad \frac{\partial c}{\partial r}=0$

$$\frac{\partial c}{\partial r}=\frac{\partial \bar{c}}{\partial r}\cdot\frac{1}{r}-\frac{1}{r^2}\bar{c} \qquad \times\, r^2$$

$$r^2\frac{\partial c}{\partial r}=r\frac{\partial \bar{c}}{\partial r}-\bar{c} \quad , \quad r=0 \Rightarrow \bar{c}=0$$

$$0=e^{-\beta^2 t}\left(A(1)+B(0)\right) \quad , \quad e^{-\beta^2 t}\neq 0 \Rightarrow A=0$$

$$\bar{c}(r,t)=e^{-\beta^2 t}\cdot B\cdot \sin\frac{\beta}{\sqrt{D}}r$$

B.C. 2 $\quad r=1 \quad c=0 \Rightarrow \bar{c}=0$

$$0=e^{-\beta^2 t}\cdot B\cdot \sin\frac{\beta}{\sqrt{D}}(1)$$

$$\sin\frac{\beta}{\sqrt{D}}=0 \Rightarrow \frac{\beta}{\sqrt{D}}=n\pi \Rightarrow \beta=n\pi\sqrt{D}$$

$$\bar{c}(r,t)=e^{-(n\pi\sqrt{D})^2\cdot t}\cdot B\cdot \sin n\pi r$$

$$\bar{c}(r,t)=\sum_{n=0}^{\infty}e^{-(n\pi\sqrt{D})^2\cdot t}\cdot B_n\cdot \sin n\pi r$$

I.C. $\quad t=0 \quad c=10 \Rightarrow \bar{c}=10r$

$$10r=\sum_{n=0}^{\infty}(1)\cdot B_n\cdot \sin n\pi r$$

$$B_n=\frac{2}{L}\int_0^L f(x)\sin nx\,dx$$

$$B_n=\frac{2}{1}\int_0^1 10r\,\sin n\pi r\,dr$$

$$B_n = 20\int_0^1 r \sin n\pi r \, dr$$

$$\int u\,dv = uv - \int v\,du \qquad u = r \qquad du = dr$$

$$dv = \sin n\pi r\,dr$$

$$B_n = 20\left[\frac{-r}{n\pi}\cos n\pi r + \frac{1}{n\pi}\int_0^1 \cos n\pi r\,dr\right] \qquad V = -\frac{1}{n\pi}\cos n\pi r$$

$$B_n = 20\left[\frac{-r}{n\pi}\cos n\pi r + \frac{1}{n^2\pi^2}\cancel{\sin n\pi r}\Big]_0^1\right]$$

$$B_n = \frac{-20r}{n\pi}\cos n\pi r$$

$$\bar{C}(r,t) = \sum_{n=0}^{\infty}\frac{-20r}{n\pi}\cos n\pi r \cdot e^{-(n\pi\sqrt{D})^2 \cdot t} \cdot \sin n\pi r$$

$$\because C = \frac{\bar{C}}{r}$$

$$\therefore C(r,t) = \frac{1}{r}\sum_{n=0}^{\infty}\frac{-20r}{n\pi}\cos n\pi r \cdot e^{-(n\pi\sqrt{D})^2 \cdot t} \cdot \sin n\pi r$$

$$C(r,t) = \sum_{n=0}^{\infty}\frac{-20}{n\pi}\cos n\pi r \cdot e^{-(n\pi\sqrt{D})^2 \cdot t} \cdot \sin n\pi r$$

Combination of Variables :

In this method we introduce a dummy variable, η, where the choice of η is given in the table below. We see that the bounded variable, e.g., distance (x, y or r) appears in the numerator raised to the power 1, while the unbounded variable such as time (t) appears in the denominator raised to the power ($1/n$), where n equals the sum of powers of the bounded variable appearing in the equation.

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} \qquad \eta = \frac{x}{\sqrt{t}}$$

$$\frac{\partial T}{\partial t} = \alpha^2\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad \eta = \frac{x+y}{\sqrt{t}}$$

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) \qquad \eta = \frac{x+y+z}{\sqrt{t}}$$

$$y\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial y^2} \qquad \eta = \frac{y}{(t)^{1/3}}$$

$$x^2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \eta = \frac{x}{(t)^{1/4}}$$

Ex: Solve the partial differential equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

i) $C(x,0) = 0$
ii) $C(0,t) = C_i$
iii) $C(\infty,t) = 0$

We start by putting, $\eta = \frac{x}{\sqrt{t}}$

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} \Rightarrow \frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta} \cdot \left[-\frac{1}{2} x \, t^{-3/2} \right]$$

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta} \left[-\frac{1}{2} x \frac{1}{\sqrt{t} \cdot t} \right] \Rightarrow \frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta} \left[-\frac{1}{2} \frac{\eta}{t} \right]$$

$$\therefore \frac{\partial c}{\partial t} = -\frac{1}{2} \frac{\eta}{t} \frac{\partial c}{\partial \eta}$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta} \left[\frac{1}{\sqrt{t}} \right] \Rightarrow \frac{\partial c}{\partial x} = \frac{1}{\sqrt{t}} \frac{\partial c}{\partial \eta}$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial c}{\partial x} \right) \Rightarrow \frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{t}} \frac{\partial c}{\partial \eta} \right)$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \left(\frac{1}{\sqrt{t}} \cdot \frac{\partial c}{\partial \eta} \right) \Rightarrow \frac{\partial^2 c}{\partial x^2} = \frac{1}{\sqrt{t}} \frac{\partial}{\partial \eta} \left(\frac{1}{\sqrt{t}} \cdot \frac{\partial c}{\partial \eta} \right)$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{1}{t} \frac{\partial^2 c}{\partial \eta^2}$$

Sub. in equation :

$$-\frac{1}{2} \frac{\eta}{t} \frac{\partial c}{\partial \eta} = D \frac{1}{t} \frac{\partial^2 c}{\partial \eta^2}$$

$$D \frac{\partial^2 c}{\partial \eta^2} + \frac{1}{2} \eta \frac{\partial c}{\partial \eta} = 0 \Rightarrow D \frac{d^2 c}{d \eta^2} + \frac{1}{2} \eta \frac{dc}{d \eta} = 0$$

This is a second order ordinary differential equation where the dependent variable is not explicit.

$$P = A e^{-\frac{1}{4}\frac{\eta^2}{D}} \Rightarrow \frac{dc}{d\eta} = A e^{-\frac{1}{4}\frac{\eta^2}{D}}$$

$$dc = A e^{-\frac{1}{4}\frac{\eta^2}{D}} . d\eta \Rightarrow C = A \int e^{-\frac{1}{4}\frac{\eta^2}{D}} d\eta$$

$$C = A \,\mathrm{erf} \sqrt{\frac{\eta^2}{4D}} + B \Rightarrow C = A \,\mathrm{erf} \frac{\eta}{\sqrt{4D}} + B$$

$$C = A \,\mathrm{erf} \frac{x}{\sqrt{4Dt}} + B$$ general solution

B.C. 1 $\quad x = 0 \quad C = C_i$

$C_i = A \,\mathrm{erf}(0) + B \quad , \quad \mathrm{erf}(0) = 0 \Rightarrow B = C_i$

B.C. 2 $\quad x = \infty \quad C = 0$

$0 = A \,\mathrm{erf}(\infty) + B \quad , \mathrm{erf}(\infty) = 1 \Rightarrow A = -B$

$\therefore A = -C_i$

$$C = -C_i \,\mathrm{erf} \frac{x}{\sqrt{4Dt}} + C_i$$

$$C = C_i \left(1 - \mathrm{erf} \frac{x}{\sqrt{4Dt}}\right)$$

$$C = C_i \,\mathrm{erfc} \frac{x}{\sqrt{4Dt}}$$

Ex: Solve the partial differential equation

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

For the conditions

i) $\theta(x,0) = 0$

ii) $\theta(0,t) = 100$

iii) $\frac{\partial \theta}{\partial x}(1,t) = 0$

We start by putting $\quad \eta = \frac{x}{2h\sqrt{t}}$

$$\frac{\partial \eta}{\partial t} = \frac{-x}{2h \cdot 2\, t^{3/2}} = \frac{-x}{4ht\sqrt{t}} = \frac{-\eta}{2t}.$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{2h\sqrt{t}}$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} \Rightarrow \frac{\partial \theta}{\partial t} = \frac{-\eta}{2t}\frac{\partial \theta}{\partial \eta}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) \Rightarrow \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}\left(\frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}\right)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{2h\sqrt{t}}\frac{\partial}{\partial \eta}\left(\frac{1}{2h\sqrt{t}}\frac{\partial \theta}{\partial \eta}\right)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{4h^2 t}\frac{\partial^2 \theta}{\partial \eta^2}$$

Sub. in equation

$$\frac{-\eta}{2t}\frac{\partial \theta}{\partial \eta} = h^2 \cdot \frac{1}{4h^2 t}\frac{\partial^2 \theta}{\partial \eta^2}$$

$$\frac{\partial^2 \theta}{\partial \eta^2} + 2\eta\frac{\partial \theta}{\partial \eta} = 0 \Rightarrow \frac{d^2 \theta}{d\eta^2} + 2\eta\frac{d\theta}{d\eta} = 0$$

This is a second order ordinary differential equation where the dependent variable is not explicit.

$$P = \frac{d\theta}{d\eta} \quad , \quad \frac{dP}{d\eta} = \frac{d^2\theta}{d\eta^2}$$

$$\frac{dP}{d\eta} + 2\eta P = 0 \Rightarrow \frac{dP}{P} + 2\eta\, d\eta = 0$$

$$\ln \frac{P}{A} = -\eta^2 \Rightarrow P = A e^{-\eta^2} \Rightarrow \frac{d\theta}{d\eta} = A e^{-\eta^2}$$

$$d\theta = A e^{-\eta^2} d\eta$$

$$\theta(x,t) = A \operatorname{erf} \eta + B$$

$$\theta(x,t) = A \operatorname{erf} \frac{x}{2h\sqrt{t}} + B \qquad \text{general solution}$$

B.C. 1 $\quad x = 0 \quad \theta = 100$

$$\theta(0,t) = A \operatorname{erf}(0) + B = 100 \quad , \operatorname{erf}(0) = 0$$

$$\therefore B = 100$$

$$\theta(x,t) = A \operatorname{erf} \frac{x}{2h\sqrt{t}} + 100$$

I.C. $\quad t = 0 \quad \theta = 0$

$$\theta(x,0) = A \operatorname{erf} \frac{x}{2h\sqrt{0}} + 100 = 0$$

$$A \operatorname{erf}(\infty) + 100 = 0 \quad , \quad \operatorname{erf}(\infty) = 1$$

$$A = -100$$

$$\theta(x,t) = -100 \operatorname{erf} \frac{x}{2h\sqrt{t}} + 100$$

$$\theta(x,t) = 100\left(1 - \text{erf}\,\frac{x}{2h\sqrt{t}}\right)$$

$$\theta(x,t) = 100\,\text{erfc}\,\frac{x}{2h\sqrt{t}}$$

Ex: Solve the partial differential equation

$$y\,\frac{\partial C_A}{\partial z} = \frac{\partial^2 C_A}{\partial y^2}$$

For the following conditions

$z = 0 \qquad C_A = 0$

$y = 0 \qquad C_A = C_{A0}$

$y = \infty \qquad C_A = 0$

We start by putting $\qquad \eta = \dfrac{y}{z^{1/3}}$

$$\frac{\partial \eta}{\partial y} = \frac{1}{z^{1/3}}$$

$$\frac{\partial \eta}{\partial z} = -\frac{1}{3}\,y\,z^{-4/3} \Rightarrow \frac{\partial \eta}{\partial z} = -\frac{y}{z^{1/3}}\,\frac{1}{3z} \Rightarrow \frac{\partial \eta}{\partial z} = -\frac{\eta}{3z}$$

$$\frac{\partial C_A}{\partial z} = \frac{\partial C_A}{\partial \eta}\,\frac{\partial \eta}{\partial z} = -\frac{\eta}{3z}\,\frac{\partial C_A}{\partial \eta}$$

$$\frac{\partial C_A}{\partial y} = \frac{\partial C_A}{\partial \eta}\,\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 C_A}{\partial y^2} = \frac{\partial}{\partial \eta}\,\frac{\partial \eta}{\partial y}\left(\frac{\partial C_A}{\partial \eta}\,\frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial^2 C_A}{\partial y^2} = \frac{\partial}{\partial \eta}\left(\frac{1}{z^{1/3}}\right)\left(\frac{\partial C_A}{\partial \eta}\left(\frac{1}{z^{1/3}}\right)\right)$$

$$\frac{\partial^2 C_A}{\partial y^2} = \frac{1}{z^{2/3}}\,\frac{\partial^2 C_A}{\partial \eta^2}$$

$$y\left(-\frac{\eta}{3z}\frac{\partial C_A}{\partial \eta}\right)=\frac{1}{z^{2/3}}\frac{\partial^2 C_A}{\partial \eta^2}$$

$$\frac{\partial^2 C_A}{\partial \eta^2}+\frac{1}{3}\eta^2\frac{\partial C_A}{\partial \eta}=0 \Rightarrow \frac{d^2 C_A}{d\eta^2}+\frac{1}{3}\eta^2\frac{dC_A}{d\eta}=0$$

$$P=\frac{dC_A}{d\eta}\quad , \quad \frac{dP}{d\eta}=\frac{d^2 C_A}{d\eta^2}$$

$$\frac{dP}{d\eta}+\frac{1}{3}\eta^2 P=0 \Rightarrow \frac{dP}{P}+\frac{1}{3}\eta^2 d\eta=0$$

$$\ln P+\frac{1}{9}\eta^3-\ln A=0 \Rightarrow P=A\,e^{-\eta^3/9}$$

$$\frac{dC_A}{d\eta}=A\,e^{-\eta^3/9} \Rightarrow \int_{C_A}^{0} dC_A=A\int_{\eta}^{\infty} e^{-\eta^3/9}\,d\eta$$

$$-C_A=A\int_{\eta}^{\infty} e^{-\eta^3/9}\,d\eta+B$$

B.C.1 $\quad y=0 \quad C_A=C_{A0} \quad \eta=0$

B.C.2 $\quad y=\infty \quad C_A=0 \quad \eta=\infty$

Apply B.C.2 $\quad \eta=\infty \quad C_A=0$

$$0=A\int_{\infty}^{\infty} e^{-\eta^3/9}\,d\eta+B \Rightarrow B=0$$

$$\therefore C_A=-A\int_{\eta}^{\infty} e^{-\eta^3/9}\,d\eta$$

Apply B.C.1 $\quad \eta=0 \quad C_A=C_{A0}$

$$C_{A0}=-A\int_{0}^{\infty} e^{-\eta^3/9}\,d\eta$$

This integration is the Gamma function (Γ).