

$$\bar{y}(s) = \frac{3s}{s^2+4} - \frac{10}{s^2+4} + \frac{4}{s^2}$$

$$y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{3s}{s^2+4} - \frac{10}{s^2+4} + \frac{4}{s^2}\right)$$

$$y = 3\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - 5\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) + 4\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$

$$y = 3\cos 2t - 5\sin 2t + 4t$$

Example: Solve the equation

$$y' + 2y = e^t, \quad y(0) = 1$$

$$\mathcal{L}(y') + \mathcal{L}(2y) = \mathcal{L}(e^t)$$

$$s\bar{y}(s) - y(0) + 2\bar{y}(s) = \frac{1}{s-1}$$

$$(s+2)\bar{y}(s) = \frac{1}{s-1} + 1$$

$$\bar{y}(s) = \frac{1}{(s-1)(s+2)} + \frac{1}{s+2}$$

$$\frac{1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} = \frac{A(s+2) + B(s-1)}{(s-1)(s+2)}$$

$$A(s+2) + B(s-1) = 1 \Rightarrow (A+B)s + 2A - B = 1$$

$$A+B=0 \quad \& \quad 2A - B = 1 \Rightarrow A = 1/3 \quad \& \quad B = -1/3$$

$$\bar{y}(s) = \frac{1/3}{s-1} - \frac{1/3}{s+2} + \frac{1}{s+2}$$

$$y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{1/3}{s-1} - \frac{1/3}{s+2} + \frac{1}{s+2}\right)$$

$$y = \frac{1}{3}e^t - \frac{1}{3}e^{-2t} + e^{-2t} \Rightarrow y = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}$$

Example: Solve the equation

$$\frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} - y = t^2 e^t.$$

For the boundary condition

$$\frac{d^2 y}{dt^2}(0) = -2, \quad \frac{dy}{dt}(0) = 0, \quad y(0) = 1$$

$$\mathcal{L}\left(\frac{d^3 y}{dt^3}\right) - \mathcal{L}\left(3 \frac{d^2 y}{dt^2}\right) + \mathcal{L}\left(3 \frac{dy}{dt}\right) - \mathcal{L}(y) = \mathcal{L}(t^2 e^t)$$

$$s^3 \bar{y}(s) - s^2 y(0) - s y'(0) - y''(0) - 3(s^2 \bar{y}(s) - s y(0) - y'(0)) + 3(s \bar{y}(s) - y(0))$$

$$- \bar{y}(s) = \frac{2!}{(s-1)^3}$$

$$s^3 \bar{y}(s) - s^2 + 2 - 3(s^2 \bar{y}(s) - s) + 3(s \bar{y}(s) - 1) - \bar{y}(s) = \frac{2!}{(s-1)^3}$$

$$(s^3 - 3s^2 + 3s - 1) \bar{y}(s) - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$

$$\bar{y}(s) = \frac{s^2 - 3s + 1}{s^3 - 3s^2 + 3s - 1} + \frac{2}{(s-1)^3 (s^3 - 3s^2 + 3s - 1)}$$

$$\bar{y}(s) = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y}(s) = \frac{s^2 - 2s + 1 - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y}(s) = \frac{(s-1)^2 - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y}(s) = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y}(s) = \frac{(s-1)^2}{(s-1)^3} - \frac{s-1}{(s-1)^3} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}\right)$$

$$y = e^t - t e^t - \frac{1}{2!} t^2 e^t + \frac{2}{5!} t^5 e^t$$

Example: Solve the equation

$$\frac{d^2 y}{dt^2} + 9y = \cos 2t$$

for the condition

$$y\left(\frac{\pi}{2}\right) = -1 \quad \& \quad y(0) = 1$$

$$\text{Let } y'(0) = C$$

$$\mathcal{L}\left(\frac{d^2 y}{dt^2}\right) + \mathcal{L}(9y) = \mathcal{L}(\cos 2t)$$

$$s^2 \bar{y}(s) - s y(0) - y'(0) + 9 \bar{y}(s) = \frac{s}{s^2 + 4}$$

$$s^2 \bar{y}(s) - s - C + 9 \bar{y}(s) = \frac{s}{s^2 + 4}$$

$$(s^2 + 9) \bar{y}(s) = s + C + \frac{s}{s^2 + 4}$$

$$\bar{y}(s) = \frac{s}{s^2 + 9} + \frac{C}{s^2 + 9} + \frac{s}{(s^2 + 4)(s^2 + 9)}$$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$(As+B)(s^2+9) + (Cs+D)(s^2+4) = s$$

$$As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs + Ds^2 + 4D = s$$

$$(A+C)s^3 + (B+D)s^2 + (9A+4C)s + (9B+4D) = s$$

$$A+C=0 \Rightarrow A=-C$$

$$B+D=0 \Rightarrow B=-D$$

$$9A+4C=1 \Rightarrow -9C+4C=1 \Rightarrow -5C=1 \Rightarrow C=-1/5$$

$$9B+4D=0 \Rightarrow -9D+4D=0 \Rightarrow -5D=0 \Rightarrow D=0$$

$$A=1/5, B=0, C=-1/5, D=0$$

$$\bar{y}(s) = \frac{s}{s^2+9} + \frac{0}{s^2+9} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}$$

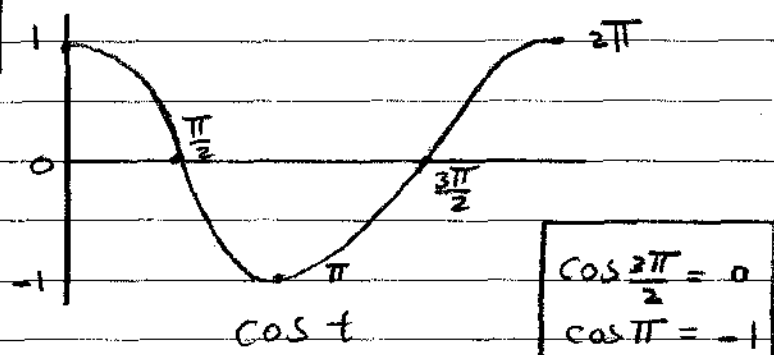
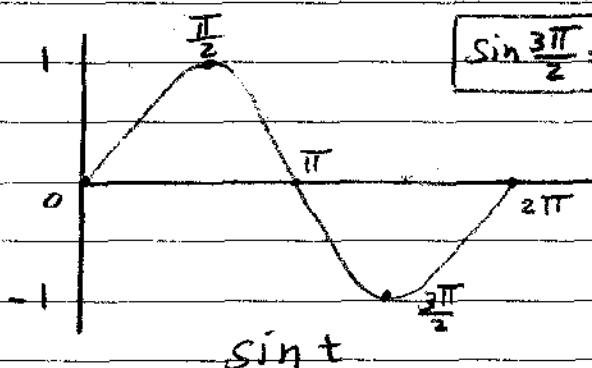
$$y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{s}{s^2+9} + \frac{0}{s^2+9} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}\right)$$

$$y = \cos 3t + \frac{0}{3} \sin 3t + \frac{1}{5} \cos 2t - \frac{1}{5} \cos 3t$$

$$\text{To find } C \Rightarrow \text{at } t = \frac{\pi}{2} \Rightarrow y = -1$$

$$-1 = 0 - \frac{C}{3} - \frac{1}{5} \Rightarrow C = \frac{12}{5}$$

$$y = \cos 3t + \frac{12}{5} \sin 3t + \frac{1}{5} \cos 2t - \frac{1}{5} \cos 3t$$



## 2. Ordinary Differential Equations with Variable Coefficients:

Example: Solve the equation

$$y'' - ty' + y = 1$$

For the Boundary Condition

$$y(0) = 1 \quad \& \quad y'(0) = 2$$

$$\mathcal{L}(y'') - \mathcal{L}(ty') + \mathcal{L}(y) = \mathcal{L}(1)$$

$$s^2 \bar{y}(s) - sy(0) - y'(0) + \frac{d}{ds}(s\bar{y}(s) - y(0)) + \bar{y}(s) = \frac{1}{s}$$

$$* \mathcal{L}(x^n f(x)) = (-1)^n \frac{d^n f(s)}{ds^n}, \quad x=t \quad \& \quad n=1$$

$$s^2 \bar{y}(s) - s - 2 + s\bar{y}'(s) + \bar{y}(s) + \bar{y}(s) = \frac{1}{s}$$

$$s\bar{y}'(s) + (s^2 + 2)\bar{y}(s) = s + 2 + \frac{1}{s} \quad \div s$$

$$\bar{y}'(s) + \left(s + \frac{2}{s}\right)\bar{y}(s) = 1 + \frac{2}{s} + \frac{1}{s^2}$$

$$\frac{d\bar{y}(s)}{ds} + \left(s + \frac{2}{s}\right)\bar{y}(s) = 1 + \frac{2}{s} + \frac{1}{s^2} \quad \text{Linear}$$

$$* \frac{dy}{dx} + P(x)y = Q(x), \quad x=s$$

$$R = e^{\int \left(s + \frac{2}{s}\right) ds} = e^{\frac{1}{2}s^2 + 2\ln s} = s^2 e^{\frac{1}{2}s^2}$$

$$* Ry = \int R \cdot Q(x) dx + C$$

$$s^2 e^{\frac{1}{2}s^2} \bar{y}(s) = \int \left(1 + \frac{2}{s} + \frac{1}{s^2}\right) s^2 e^{\frac{1}{2}s^2} ds + C$$

$$\bar{y}(s) = \frac{1}{s^2} e^{-\frac{1}{2}s^2} \int (1 + \frac{2}{s} + \frac{1}{s^2}) s^2 e^{\frac{1}{2}s^2} ds + C$$

$$\bar{y}(s) = \frac{1}{s^2} e^{-\frac{1}{2}s^2} \int (s^2 e^{\frac{1}{2}s^2} + 2s e^{\frac{1}{2}s^2} + e^{\frac{1}{2}s^2}) ds + C$$

$$u = s \quad du = ds$$

$$dv = s e^{\frac{1}{2}s^2} ds \quad v = e^{\frac{1}{2}s^2}$$

$$\bar{y}(s) = \frac{1}{s^2} e^{-\frac{1}{2}s^2} \left( s e^{\frac{1}{2}s^2} \int e^{\frac{1}{2}s^2} ds + 2 e^{\frac{1}{2}s^2} + \int e^{\frac{1}{2}s^2} ds \right) + C$$

$$\bar{y}(s) = \frac{1}{s^2} e^{-\frac{1}{2}s^2} \left( s e^{\frac{1}{2}s^2} + 2 e^{\frac{1}{2}s^2} + C \right)$$

$$\bar{y}(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{C}{s^2} e^{-\frac{1}{2}s^2}$$

$$\bar{y}(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{C}{s^2} \left( 1 - \frac{1}{2}s^2 + \frac{1}{8}s^4 - \dots \right)$$

$$\bar{y}(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{C}{s^2} - \frac{C}{2} + \frac{C}{8}s^2 - \dots$$

$$\bar{y}(s) = \frac{1}{s} + \frac{C+2}{s^2} - C \left( \frac{1}{2} - \frac{1}{8}s^2 + \dots \right)$$

$$* \mathcal{L}^{-1}(s^x) = 0 \quad \text{for } x = 0, 1, 2, 3, \dots$$

$$\mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1} \left( \frac{1}{s} + \frac{C+2}{s^2} - C \left( \frac{1}{2} - \frac{1}{8}s^2 + \dots \right) \right)$$

$$y = 1 + (C+2)t \Rightarrow y' = 0 + C+2$$

$$y'(0) = 2 \Rightarrow 2 = 0 + C+2 \Rightarrow C = 0$$

$$\therefore y = 1 + 2t$$

Example: Solve the equation

$$ty'' - ty' + y = 0$$

For the boundary condition

$$y(0) = 0 \quad \& \quad y'(0) = 1$$

$$\mathcal{L}(ty'') - \mathcal{L}(ty') + \mathcal{L}(y) = 0$$

$$-\frac{d}{ds}(s^2 \bar{y}(s) - sy(0) - y'(0)) + \frac{d}{ds}(s \bar{y}(s) - y(0)) + \bar{y}(s) = 0$$

$$-\frac{d}{ds}(s^2 \bar{y}(s) - 1) + \frac{d}{ds}(s \bar{y}(s)) + \bar{y}(s) = 0$$

$$-s^2 \bar{y}'(s) - 2s \bar{y}(s) + s \bar{y}'(s) + \bar{y}(s) + \bar{y}(s) = 0$$

$$(-s^2 + s) \bar{y}'(s) + (-2s + 1 + 1) \bar{y}(s) = 0$$

$$-s(s-1) \bar{y}'(s) - 2(s-1) \bar{y}(s) = 0$$

$$s \bar{y}'(s) + 2 \bar{y}(s) = 0 \Rightarrow s \frac{d\bar{y}(s)}{ds} + 2 \bar{y}(s) = 0$$

$$\frac{d\bar{y}(s)}{2 \bar{y}(s)} = -\frac{ds}{s} \Rightarrow \frac{1}{2} \ln \bar{y}(s) = -\ln s + \ln c$$

$$\bar{y}(s) = \frac{k}{s^2} \Rightarrow y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{k}{s^2}\right) = kt$$

$$y = kt \Rightarrow y' = k \quad \& \quad y'(0) = 1 \Rightarrow k = 1$$

$$\therefore y = t$$

Example: Solve the equation

$$ty'' - ty' + y = 0$$

For the boundary condition

$$y(0) = 0 \quad \& \quad y'(0) = 1$$

$$\mathcal{L}(ty'') - \mathcal{L}(ty') + \mathcal{L}(y) = 0$$

$$-\frac{d}{ds}(s^2 \bar{y}(s) - sy(0) - y'(0)) + \frac{d}{ds}(s\bar{y}(s) - y(0)) + \bar{y}(s) = 0$$

$$-\frac{d}{ds}(s^2 \bar{y}(s) - 1) + \frac{d}{ds}(s\bar{y}(s)) + \bar{y}(s) = 0$$

$$-s^2 \bar{y}'(s) - 2s\bar{y}(s) + s\bar{y}'(s) + \bar{y}(s) + \bar{y}(s) = 0$$

$$(-s^2 + s) \bar{y}'(s) + (-2s + 1 + 1) \bar{y}(s) = 0$$

$$-s(s-1) \bar{y}'(s) - 2(s-1) \bar{y}(s) = 0$$

$$s \bar{y}'(s) + 2 \bar{y}(s) = 0 \Rightarrow s \frac{d\bar{y}(s)}{ds} + 2 \bar{y}(s) = 0$$

$$\frac{d\bar{y}(s)}{2 \bar{y}(s)} = -\frac{ds}{s} \Rightarrow \frac{1}{2} \ln \bar{y}(s) = -\ln s + \ln C$$

$$\bar{y}(s) = \frac{K}{s^2} \Rightarrow y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{K}{s^2}\right) = Kt$$

$$y = Kt \Rightarrow y' = K \quad \& \quad y'(0) = 1 \Rightarrow K = 1$$

$$\therefore y = t$$



## Partial Differential Equations by Laplace Transformation:

Example: Solve the PDE by using:

1. Separation of variable method.
2. Laplace transform method.

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial^2 \vartheta}{\partial x^2}$$

For the boundary condition

i)  $\vartheta(0, t) = 0$

ii)  $\vartheta(1, t) = 0$

iii)  $\vartheta(x, 0) = 3 \sin 2\pi x$

1. Separation of variable:

$$\vartheta(x, t) = e^{-\beta^2 t} (A \cos \beta x + B \sin \beta x)$$

B.C. 1  $x=0, \vartheta=0$

$$0 = e^{-\beta^2 t} (A(1) + B(0)) \Rightarrow A=0$$

$$\vartheta(x, t) = B e^{-\beta^2 t} \sin \beta x$$

B.C. 2  $x=1, \vartheta=0$

$$0 = B e^{-\beta^2 t} \sin \beta(1) \Rightarrow \beta = n\pi$$

$$\vartheta(x, t) = B e^{-(n\pi)^2 t} \sin n\pi x$$

I.C.  $t=0, \vartheta = 3 \sin 2\pi x$

$$3 \sin 2\pi x = B \sin n\pi x$$

$$\therefore B = 3 \quad \& \quad n = 2$$

$$\vartheta(x,t) = 3 e^{-4\pi^2 t} \sin 2\pi x$$

2. Laplace transform:

$$\mathcal{L} \frac{\partial \vartheta}{\partial t} = \mathcal{L} \frac{\partial^2 \vartheta}{\partial x^2}$$

$$s \bar{\vartheta}(s) - \vartheta(0) = \frac{d^2 \bar{\vartheta}(s)}{dx^2}$$

$$s \bar{\vartheta}(s) - 3 \sin 2\pi x = \frac{d^2 \bar{\vartheta}(s)}{dx^2}$$

$$\frac{d^2 \bar{\vartheta}(s)}{dx^2} - s \bar{\vartheta}(s) = -3 \sin 2\pi x$$

$$(D^2 - s) \bar{\vartheta}(s) = -3 \sin 2\pi x$$

$$m^2 - s = 0 \Rightarrow m = \pm \sqrt{s}$$

$$\bar{\vartheta}(s)_c = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x}$$

$$\bar{\vartheta}(s)_p = \frac{-3 \sin 2\pi x}{D^2 - s}$$

$$y_p = \frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \sin(ax+b)$$

$$\bar{\vartheta}(s)_p = \frac{-3 \sin 2\pi x}{-(2\pi)^2 - s}$$

$$\bar{\vartheta}(s)_p = \frac{3 \sin 2\pi x}{4\pi^2 + s}$$

$$\bar{Q}(s) = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x} + \frac{3 \sin 2\pi x}{s + 4\pi^2}$$

B.C.1  $x=0$ ,  $Q=0 \Rightarrow \bar{Q}(s)=0$

$$0 = C_1(1) + C_2(1) + 0, \quad \sin(0) = 0$$

$$\therefore C_1 = -C_2$$

B.C.2  $x=1$ ,  $Q=0 \Rightarrow \bar{Q}(s)=0$

$$0 = C_1 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}} + 0, \quad \sin(2\pi) = 0$$

$$0 = -C_2 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}}$$

$$0 = C_2 (-e^{\sqrt{s}} + e^{-\sqrt{s}}) \Rightarrow C_2 = 0$$

$$C_1 = -C_2 \Rightarrow C_1 = 0$$

$$\therefore \bar{Q}(s) = \frac{3 \sin 2\pi x}{s + 4\pi^2}$$

$$\mathcal{L}^{-1} \bar{Q}(s) = 3 \sin 2\pi x \mathcal{L}^{-1} \frac{1}{s + 4\pi^2}$$

$$Q(x,t) = 3 \sin 2\pi x e^{-4\pi^2 t}$$