$$\int_{100}^{M} dM = \int_{0}^{t} dt \Rightarrow M\Big]_{100}^{M} = t\Big]_{0}^{t} \Rightarrow M - 100 = t - 0$$

$$M = 100 + t$$

Solute material balance

In = Out + Accumulation

$$W(0) + S(1) = B(x) + \frac{d(Mx)}{dt}$$

$$1 = 10x + x\frac{dM}{dt} + M\frac{dx}{dt}$$

$$1 = 10x + x(1) + (100+t)\frac{dx}{dt}$$

$$1 = 11x + (100+t)\frac{dx}{dt} \Rightarrow (1-11x) = (100+t)\frac{dx}{dt}$$

$$\int \frac{dx}{(1-11x)} = \int \frac{dt}{(100+t)}$$

At $t = 0$ $x = 0$

At $t = t$ $x = x$

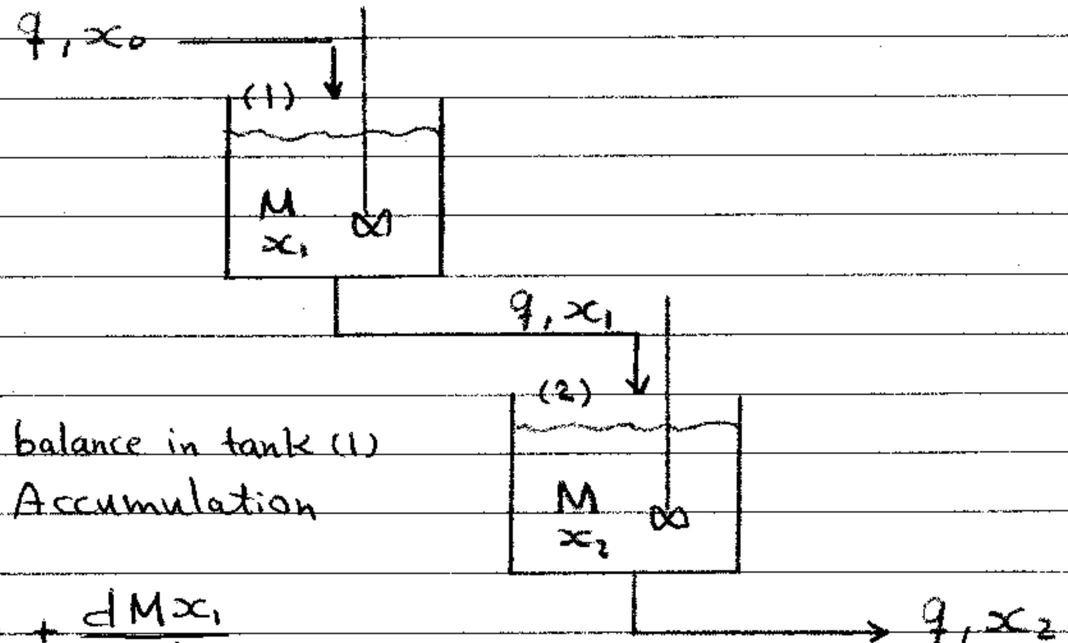
$$\int_{0}^{x} \frac{dx}{(1-11x)} = \int_{0}^{t} \frac{dt}{(100+t)}$$

$$-\frac{1}{11}\ln(1-11x)\Big]_{0}^{x} = \ln(100+t)\Big]_{0}^{t}$$

$$\frac{1}{11}\ln\left(\frac{1}{1-11x}\right)\Big]_{0}^{x} = \ln(100+t)\Big]_{0}^{t}$$

$$\ln\left(\frac{1}{1-11x}\right)^{1/11} = \ln\left(\frac{100+t}{100}\right) \Rightarrow \left(\frac{1}{1-11x}\right)^{1/11} = \frac{100+t}{100}$$

Example: Two mixer are connected in series, each of them contain M kg of water. Intitially, q (Kg/hr) of water flows to the first mixer containing solute with x_0. Find the concentration in the second mixer when a step change Δx_0 is take place in the inlet stream to mixer (1).



Solute material balance in tank (1)

In = Out + Accumulation

$$q x_0 = q x_1 + \frac{d M x_1}{dt}$$

$$q x_0 = q x_1 + M \frac{dx_1}{dt}$$

$$\frac{M}{q}\frac{dx_1}{dt} + x_1 = x_0 \Rightarrow \tau \frac{dx_1}{dt} + x_1 = x_0$$

$$\tau \frac{dx_1'}{dt} + x_1' = \Delta x_0 \qquad \text{Taking Laplace Transform}$$

$$\tau\left(s\bar{x}_1'(s) - x_1'(0)\right) + \bar{x}_1'(s) = \frac{\Delta x_0}{s}$$

At $t=0$ $x_1 = 0$ or $x_1' = 0$

$$\tau(s\bar{x}_1'(s) - 0) + \bar{x}_1'(s) = \frac{\Delta x_0}{s}$$

$$(\tau s + 1)\bar{x}_1'(s) = \frac{\Delta x_0}{s} \Rightarrow \bar{x}_1'(s) = \frac{\Delta x_0}{s(\tau s + 1)} \quad \dots(1)$$

Solute material balance in tank (2)

In = Out + Accumulation

$$q x_1 = q x_2 + \frac{d M x_2}{dt} \Rightarrow q x_1 = q x_2 + M \frac{dx_2}{dt}$$

$$\frac{M}{q} \frac{dx_2}{dt} + x_2 = x_1 \Rightarrow \tau \frac{dx_2}{dt} + x_2 = x_1$$

$$\tau \frac{dx'_2}{dt} + x'_2 = x'_1$$ Taking Laplace Transform

$$\tau (s \bar{x}'_2(s) - x'_2(0)) + \bar{x}'_2(s) = \bar{x}'_1(s)$$

At $t=0$ $x_2 = 0$ or $x'_2 = 0$

$$(\tau s + 1) \bar{x}'_2(s) = \bar{x}'_1(s) \Rightarrow \bar{x}'_2(s) = \frac{\bar{x}'_1(s)}{\tau s + 1} \quad \ldots (2)$$

Substitute equation (1) in uation (2)

$$\bar{x}'_2(s) = \frac{\Delta x_0}{s(\tau s + 1)^2}$$

$$\frac{1}{s(\tau s+1)^2} = \frac{A}{s} + \frac{B}{\tau s+1} + \frac{C}{(\tau s+1)^2}$$

$$1 = A(\tau s+1)^2 + B s(\tau s+1) + C s$$
$$1 = A\tau^2 s^2 + 2A\tau s + A + B\tau s^2 + Bs + Cs$$
$$1 = (A\tau^2 + B\tau) s^2 + (2A\tau + B + C) s + A$$
$$A = 1 \quad , \quad A\tau^2 + B\tau = 0 \quad , \quad 2A\tau + B + C = 0$$
$$\therefore B = -\tau \quad \& \quad C = -\tau$$

$$\frac{1}{s(\tau s+1)^2} = \frac{1}{s} - \frac{\tau}{\tau s+1} - \frac{\tau}{(\tau s+1)^2}$$

$$\frac{1}{s(\tau s+1)^2} = \frac{1}{s} - \frac{\tau}{\tau(s+\frac{1}{\tau})} - \frac{\tau}{\tau^2(s+\frac{1}{\tau})^2}$$

$$\frac{1}{s(\tau s+1)^2} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}} - \frac{1}{\tau(s+\frac{1}{\tau})^2}$$

Taking Inverse Laplace Transform

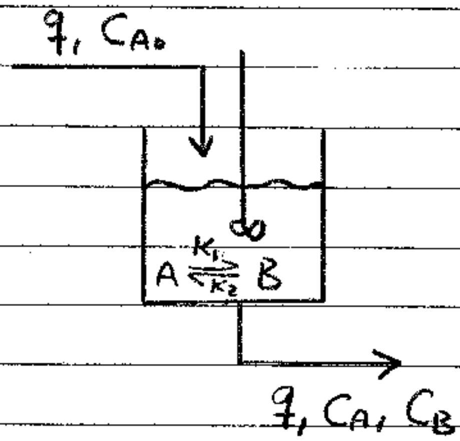
$$x_2'(t) = \Delta x_0 \left(1 - e^{-\frac{t}{\tau}} - \frac{t}{\tau} e^{-\frac{t}{\tau}}\right)$$

$$x_2(t) - x_2(0) = \Delta x_0 \left(1 - e^{-\frac{t}{\tau}} - \frac{t}{\tau} e^{-\frac{t}{\tau}}\right)$$

Example: The first order reversible reaction $A \underset{K_2}{\overset{K_1}{\rightleftharpoons}} B$ occur in continuous stirred tank reactor. Find the differential equation which relate C_A with time?

Material Balance on A

In + Generation = Out + Consumption + Accⁿ.



$$qC_{A0} + K_2 C_B V = qC_A + K_1 C_A V + \frac{d C_A V}{dt}$$

$$V\frac{dC_A}{dt} + (q + K_1 V)\, C_A = qC_{A0} + K_2 C_B V$$

$$\frac{V}{q+K_1V}\frac{dC_A}{dt} + C_A = \frac{q}{q+K_1V}C_{A0} + \frac{K_2V}{q+K_1V}C_B$$

$$\tau_1 \frac{dC_A}{dt} + C_A = C_1 C_{A0} + C_2 C_B \quad \ldots\ldots (1)$$

Material Balance on B

In + Generation = Out + Consumption + Accumulation

$$0 + K_1 C_A V = qC_B + K_2 C_B V + \frac{d C_B V}{dt}$$

$$V\frac{dC_B}{dt} + (q + K_2V)\, C_B = K_1 C_A V$$

$$\frac{V}{q+K_2V}\frac{dC_B}{dt} + C_B = \frac{K_1V}{q+K_2V}C_A$$

$$\tau_2 \frac{dC_B}{dt} + C_B = C_3 C_A \quad \ldots\ldots (2)$$

Divided equation (1) by C_2

$$\frac{\tau_1}{C_2}\frac{dC_A}{dt} + \frac{1}{C_2}C_A - \frac{C_1}{C_2}C_{A0} = C_B \quad \ldots\ldots (3)$$

Differentiate with respect to t

$$\frac{\tau_1}{C_2}\frac{d^2C_A}{dt^2}+\frac{1}{C_2}\frac{dC_A}{dt}-\frac{C_1}{C_2}(0)=\frac{dC_B}{dt} \quad \ldots\ldots (4)$$

Substitute equations (3) & (4) in equation (2)

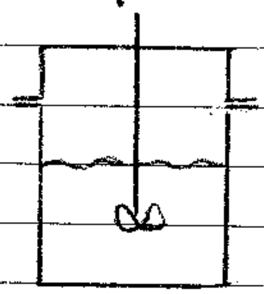
$$\tau_2\left[\frac{\tau_1}{C_2}\frac{d^2C_A}{dt^2}+\frac{1}{C_2}\frac{dC_A}{dt}\right]+\frac{\tau_1}{C_2}\frac{dC_A}{dt}+\frac{1}{C_2}C_A-\frac{C_1}{C_2}C_{A_0}=C_3C_A$$

$$\frac{\tau_1\tau_2}{C_2}\frac{d^2C_A}{dt^2}+\frac{\tau_2}{C_2}\frac{dC_A}{dt}+\frac{\tau_1}{C_2}\frac{dC_A}{dt}+\frac{1}{C_2}C_A-\frac{C_1}{C_2}C_{A_0}=C_3C_A$$

$$\frac{\tau_1\tau_2}{C_2}\frac{d^2C_A}{dt^2}+\left(\frac{\tau_1+\tau_2}{C_2}\right)\frac{dC_A}{dt}+\left(\frac{1}{C_2}-C_3\right)C_A=\frac{C_1}{C_2}C_{A_0}$$

$$\frac{d^2C_A}{dt^2}+\left(\frac{\tau_1+\tau_2}{\tau_1\tau_2}\right)\frac{dC_A}{dt}+\left(\frac{1-C_2C_3}{\tau_1\tau_2}\right)C_A=\frac{C_1}{\tau_1\tau_2}C_{A_0}$$

Example: The first order reversible reaction $A \underset{K_2}{\overset{K_1}{\rightleftharpoons}} B$ occur in batch reactor. Find the differential equation which relate CA with time?



Material Balance on A

In + Generation = Out + Consumption + Accumulation

$$0+K_2C_BV=0+K_1C_AV+V\frac{dC_A}{dt}$$

$$K_2\cancel{V}C_B=K_1\cancel{V}C_A+\cancel{V}\frac{dC_A}{dt}$$

$$K_2C_B=K_1C_A+\frac{dC_A}{dt} \quad \ldots\ldots (1)$$

Material Balance on B

In + Generation = Out + Consumption + Accumulation

$$0 + K_1 V C_A = 0 + K_2 V C_B + V \frac{dC_B}{dt}$$

$$K_1 \cancel{V} C_A = K_2 \cancel{V} C_B + \cancel{V} \frac{dC_B}{dt}$$

$$K_1 C_A = K_2 C_B + \frac{dC_B}{dt} \quad \ldots \quad (2)$$

From equation (1)

$$\frac{dC_A}{dt} + K_1 C_A = K_2 C_B \Rightarrow (D + K_1) C_A = K_2 C_B$$

$$C_B = \frac{(D + K_1)}{K_2} C_A$$

From equation (2)

$$\frac{dC_B}{dt} + K_2 C_B = K_1 C_A \Rightarrow (D + K_2) C_B = K_1 C_A$$

$$(D + K_2) \frac{(D + K_1)}{K_2} C_A = K_1 C_A$$

$$(D + K_2)(D + K_1) C_A = K_1 K_2 C_A$$
$$(D^2 + K_2 D + K_1 D + K_1 K_2) C_A = K_1 K_2 C_A$$
$$(D^2 + (K_1 + K_2) D + K_1 K_2) C_A = K_1 K_2 C_A$$

$$\frac{d^2 C_A}{dt^2} + (K_1 + K_2) \frac{dC_A}{dt} + K_1 K_2 C_A = K_1 K_2 C_A$$

$$\frac{d^2 C_A}{dt^2} + (K_1 + K_2) \frac{dC_A}{dt} = 0 \quad , \quad \lambda = K_1 + K_2$$

$$D^2 + \lambda D = 0$$
$$D (D + \lambda) = 0 \Rightarrow D_1 = 0 \; , \; D_2 = -\lambda$$

$$C_A = C_1 e^{0t} + C_2 e^{-\lambda t} \Rightarrow C_A = C_1 + C_2 e^{-\lambda t}$$

At $t=0$ $C_A = C_{A_0}$

$t=\infty$ $C_A = C_{Ae}$

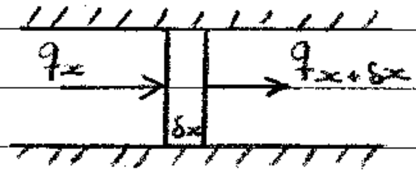
B.C (1) $\Rightarrow$ $C_{A_0} = C_1 + C_2$

B.C (2) $\Rightarrow$ $C_{Ae} = C_1 + 0 \Rightarrow C_{Ae} = C_1$

$\therefore C_2 = C_{A_0} - C_{Ae}$

$$C_A = C_1 + C_2 e^{-\lambda t} \Rightarrow C_A = C_{Ae} + (C_{A_0} - C_{Ae})\, e^{-(K_1+K_2)t}$$

Example: A hot liquid flow through a pipe (insulated) with constant velocity u. Find the differential equation which describe the variation of liquid temperature in axial distance with time?



Heat Balance

In = Out + Accumulation

$$q_x = q_{x+\delta x} + M C_p \frac{\partial T}{\partial t}$$

$$\cancel{q_x} = \cancel{q_x} + \frac{\partial q_x}{\partial x}\delta x + M C_p \frac{\partial T}{\partial t}$$

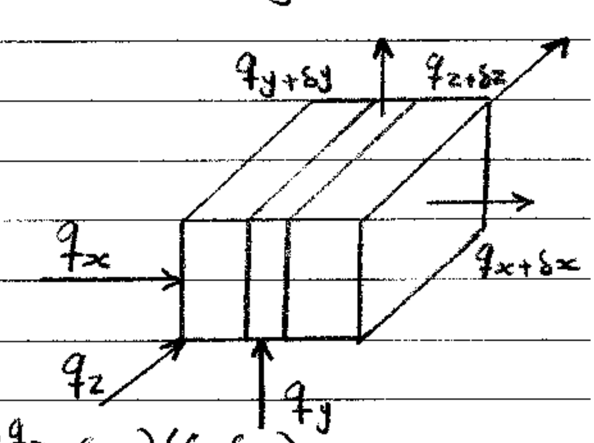
$$0 = \frac{\partial q_x}{\partial x}\delta x + M C_p \frac{\partial T}{\partial t}$$

$$0 = \dot{m} C_p \frac{\partial T}{\partial x}\delta x + \rho V C_p \frac{\partial T}{\partial t}$$

$$0 = \rho u A C_p \frac{\partial T}{\partial x}\delta x + \rho A \delta x C_p \frac{\partial T}{\partial t}$$

$$0 = u\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t} \Rightarrow \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0$$

Example: Derive the three dimensional unsteady state conduction equation?



In = Out + Accumulation

$$q_x \delta y \delta z + q_y \delta x \delta z + q_z \delta x \delta y =$$
$$q_{x+\delta x}(\delta y \delta z) + q_{y+\delta y}(\delta x \delta z) +$$
$$q_{z+\delta z}(\delta x \delta y) + M C_p \frac{\partial T}{\partial t}$$

$$q_x \delta y \delta z + q_y \delta x \delta z + q_z \delta x \delta y = \left(q_x + \frac{\partial q_x}{\partial x}\delta x\right)(\delta y \delta z) +$$

$$\left(q_y + \frac{\partial q_y}{\partial y}\delta y\right)(\delta x \delta z) + \left(q_z + \frac{\partial q_z}{\partial z}\delta z\right)(\delta x \delta y) + \overbrace{\delta x \delta y \delta z}^{V} \rho C_p \frac{\partial T}{\partial t}$$

$$0 = \frac{\partial q_x}{\partial x}\delta x \delta y \delta z + \frac{\partial q_y}{\partial y}\delta x \delta y \delta z + \frac{\partial q_z}{\partial z}\delta x \delta y \delta z + \delta x \delta y \delta z \rho C_p \frac{\partial T}{\partial t}$$

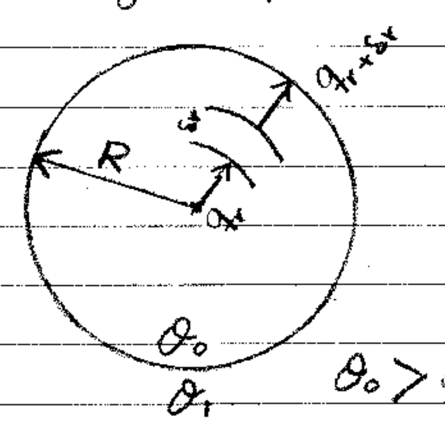
$$0 = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} + \rho C_p \frac{\partial T}{\partial t}$$

$$q_x = -K\frac{\partial T}{\partial x}, \quad q_y = -K\frac{\partial T}{\partial y}, \quad q_z = -K\frac{\partial T}{\partial z}$$

$$0 = -K\frac{\partial^2 T}{\partial x^2} - K\frac{\partial^2 T}{\partial y^2} - K\frac{\partial^2 T}{\partial z^2} + \rho C_p \frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t} = \frac{K}{\rho C_p}\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

Example: Derive heat transfer equation through a spherical body?



In = Out + Accumulation

$$q_r(4\pi r^2) = q_{r+\delta r}\left(4\pi (r+\delta r)^2\right) + M C_p \frac{\partial \theta}{\partial t}$$

$\theta_o > \theta_1$

$$q_r(4\pi r^2) = (q_r + \frac{\partial q_r}{\partial r}\delta r)(4\pi r^2 + 8\pi r\delta r + 4\pi \delta r^2) + 4\pi r^2 \delta r$$

$$\rho C_p \frac{\partial \theta}{\partial t}$$

$$q_r(4\pi r^2) = q_r(4\pi r^2) + q_r(8\pi r\delta r) + q_r(4\pi \delta r^2) + \frac{\partial q_r}{\partial r}\delta r$$

$$(4\pi r^2) + \frac{\partial q_r}{\partial r}\delta r^2(8\pi r) + \frac{\partial q_r}{\partial r}\delta r^3(4\pi) + 4\pi r^2 \delta r \rho C_p \frac{\partial \theta}{\partial t}$$

δr is small, δr^2 & δr^3 are very small $\Rightarrow$ neglected

$$0 = 2q_r + r\frac{\partial q_r}{\partial r} + r\rho C_p \frac{\partial \theta}{\partial t} \qquad \div r$$

$$0 = \frac{2}{r} q_r + \frac{\partial q_r}{\partial r} + \rho C_p \frac{\partial \theta}{\partial t}$$

$$q_r = -K\frac{\partial \theta}{\partial r}$$

$$0 = \frac{2}{r}\left(-K\frac{\partial \theta}{\partial r}\right) - K\frac{\partial^2 \theta}{\partial r^2} + \rho C_p \frac{\partial \theta}{\partial t}$$

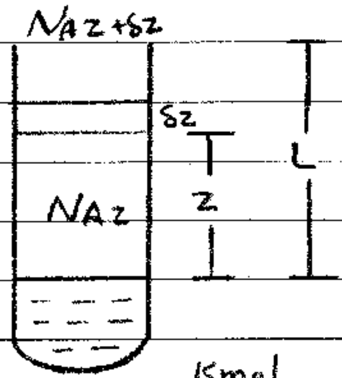
$$\frac{\partial \theta}{\partial t} = \frac{K}{\rho C_p}\left[\frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r}\frac{\partial \theta}{\partial r}\right]$$

$t = 0 \qquad \theta = \theta_0$

$r = 0 \qquad \frac{\partial \theta}{\partial r} = 0$

$r = R \qquad \theta = \theta_1$

Example: A glass tube of cross sectional (S) is filled with a volatile liquid to a certain level. The level is kept constant. It's open end is subjected to a stream air. Find the equation describing the rate of diffusion of the vapor of volatile liquid.



$N_A = \frac{Kmol}{m^2 . s}$

Mass Balance

In = Out + Accumulation

$$N_A . S = \left[N_A + \frac{\partial N_A}{\partial z}\delta z\right] S + \overset{V}{S\,\delta z}\,\frac{\partial C_A}{\partial t}$$

$$0 = \frac{\partial N_A}{\partial z}\delta z . S + \delta z\, S\,\frac{\partial C_A}{\partial t}$$

$$0 = \frac{\partial N_A}{\partial z} + \frac{\partial C_A}{\partial t} \Rightarrow \frac{\partial C_A}{\partial t} = -\frac{\partial N_A}{\partial z}$$

$$N_{A_z} = -D_A \frac{\partial C_A}{\partial z} + u_z C_A \qquad \text{Fick's law}$$

$u_z C_A$ is neglected $\Rightarrow N_A = -D_A \dfrac{\partial C_A}{\partial z}$

$$\frac{\partial C_A}{\partial t} = D_A \frac{\partial^2 C_A}{\partial z^2}$$

$t = 0$	$C_A = 0$	$t = 0$	$C = 0$
$z = 0$	$C_A = C_A^*$	$x = 0$	$C = C_i$
$z = L$	$C_A = 0$	$x = \infty$	$C = 0$