3.

$$\boxed{\mathbf{Z\{(-1)^n\} = \frac{z}{z+1}}}$$

$Z\{(-1)^n\} = \frac{z}{z+1}$ Putting $a = -1$ in Resu

4.

$$\boxed{\mathbf{Z\{k\} = \frac{kz}{z-1}}}$$

$$Z\{k\} = \sum_{n=0}^{\infty} kz^{-n} = k\sum_{n=0}^{\infty} z^{-n}$$

$$= k\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots + \frac{1}{z^n} + \cdots\right]$$

$$\therefore Z\{k\} = \frac{kz}{z-1}$$

5. Recurrence formula for n^p: $$\boldsymbol{Z\{n^p\} = -z\frac{d}{dz}Z\{n^{p-1}\}}$$

$Z\{n^p\} = \sum_{n=0}^{\infty} n^p z^{-n}$, p is a positive integer ... ①

$Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$ ... ②

Differentiating ② w.r.t. z, we get

$$\frac{d}{dz}Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1}(-n)z^{-n-1}$$

$$= -z^{-1}\sum_{n=0}^{\infty} n^p z^{-n}$$

$\Rightarrow \frac{d}{dz}Z\{n^{p-1}\} = -z^{-1}Z\{n^p\}$ using①

$\Rightarrow Z\{n^p\} = -z\frac{d}{dz}Z\{n^{p-1}\}$

6. Multiplication by n: $$\boldsymbol{Z\{nu_n\} = -z\frac{d}{dz}Z\{u_n\}}$$

$$Z\{nu_n\} = \sum_{n=0}^{\infty} nu_n z^{-n}$$

$$= -z\sum_{n=0}^{\infty} u_n(-n)z^{-n-1}$$

$$= -z\sum_{n=0}^{\infty} u_n \frac{d}{dz} z^{-n}$$

$$= -z\sum_{n=0}^{\infty} \frac{d}{dz}(u_n z^{-n})$$

$$= -z\frac{d}{dz}\left(\sum_{n=0}^{\infty} u_n z^{-n}\right)$$

$$= -z\frac{d}{dz}Z\{u_n\}$$

7. $$\boxed{\boldsymbol{Z\{n\} = \frac{z}{(z-1)^2}}}$$

$Z\{n\} = -z\frac{d}{dz}Z\{n^0\}$ using Recurrence Result **5 or 6**

$= -z\frac{d}{dz}Z\{1\}$

$= -z\frac{d}{dz}\frac{z}{z-1}$ using result **2**

$\Rightarrow Z\{n\} = \frac{z}{(z-1)^2}$

8. $$\boxed{\boldsymbol{Z\{n^2\} = \frac{z^2+z}{(z-1)^3}}}$$

$Z\{n^2\} = -z\frac{d}{dz}Z\{n\}$ using Recurrence Result **5 or 6**

$= -z\frac{d}{dz}\frac{z}{(z-1)^2}$ using Result **7**

$\Rightarrow Z\{n^2\} = \frac{z^2+z}{(z-1)^3}$

9. $$\boxed{\boldsymbol{Z\left\{u(n) = \begin{Bmatrix} 0, n < 0 \\ 1, n \geq 0 \end{Bmatrix}\right\} = \frac{z}{z-1}}}$$ $\boldsymbol{u(n) = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}}$ **is Unit step sequence**

$Z\{u(n)\} = \sum_{n=0}^{\infty} u(n)z^{-n} = \sum_{n=0}^{\infty} 1z^{-n}$

$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots + \frac{1}{z^n} + \cdots$

$\Rightarrow Z\{u(n)\} = \frac{z}{z-1}$

10. $Z\left\{\delta(n) = \begin{cases}1, n=0\\0, n\neq 0\end{cases}\right\} = 1$ $\quad \delta(n) = \begin{cases}1, n=0\\0, n\neq 0\end{cases}$ **is Unit impulse sequence**

$$Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n) z^{-n}$$
$$= 1 + 0 + 0 + \cdots$$
$$\Rightarrow Z\{\delta(n)\} = 1$$

Properties of Z-Transforms

1. **Linearity:** $Z\{au_n + bv_n\} = aZ\{u_n\} + bZ\{v_n\}$

   Proof: $Z\{au_n + bv_n\} = \sum_{n=0}^{\infty}(au_n + bv_n)z^{-n}$

   $$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n}$$
   $$= aZ\{u_n\} + bZ\{v_n\}$$

2. **Change of scale (or Damping rule):**

   If $Z\{u_n\} \equiv U(z)$, then $Z\{a^{-n}u_n\} \equiv U(az)$ and $Z\{a^n u_n\} \equiv U\left(\frac{z}{a}\right)$

   Proof: $Z\{a^{-n}u_n\} = \sum_{n=0}^{\infty} a^{-n} u_n z^{-n}$

   $$= \sum_{n=0}^{\infty} u_n (az)^{-n} \equiv U(az)$$

   Similarly $Z\{a^n u_n\} \equiv U\left(\frac{z}{a}\right)$

Results from application of Damping rule

i. $$\boldsymbol{Z\{a^n n\} = \frac{az}{(z-a)^2}}$$

Proof: $Z\{n\} = \frac{z}{(z-1)^2} \equiv U(z)$ say

$$\therefore Z\{a^n n\} \equiv U\left(\frac{z}{a}\right) = \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{az}{(z-a)^2}$$

ii. $$\boldsymbol{Z\{a^n n^2\} = \frac{az^2 + a^2 z}{(z-a)^3}}$$

Proof: $Z\{n^2\} = \frac{z^2+z}{(z-1)^3} \equiv U(z)$ say

$$\therefore Z\{a^n n^2\} \equiv U\left(\frac{z}{a}\right) = \frac{\left(\frac{z}{a}\right)^2+\left(\frac{z}{a}\right)}{\left(\left(\frac{z}{a}\right)-1\right)^3} = \frac{a(z^2+az)}{(z-a)^3}$$

iii. $$Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}, \quad Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Proof: We have $Z\left\{e^{-in\theta}\right\} = Z\left\{\left(e^{i\theta}\right)^{-n}\right\} = Z\left\{\left(e^{i\theta}\right)^{-n}.1\right\}$

Now $Z\{1\} = \frac{z}{z-1}$

$$\therefore Z\left\{\left(e^{i\theta}\right)^{-n}.1\right\} = \frac{ze^{i\theta}}{ze^{i\theta}-1} \qquad \because Z\{a^{-n}u_n\} \equiv U(az)$$

$$= \frac{z}{z-e^{-i\theta}}$$

$$= \frac{z(z-e^{i\theta})}{\left(z-e^{-i\theta}\right)\left(z-e^{i\theta}\right)}$$

$$= \frac{z\,(z-\cos\theta-i\sin\theta)}{z^2-z\left(e^{i\theta}+e^{-i\theta}\right)+1} \quad \because e^{i\theta} = \cos\theta + i\sin\theta$$

$$= \frac{z\,(z-\cos\theta-i\sin\theta}{z^2-2z\cos\theta+1} \qquad \because \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\therefore Z\left\{e^{-in\theta}\right\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - i\,\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$\Rightarrow Z\{\cos n\theta - i\sin n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - i\,\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$\therefore Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} \qquad \ldots③$$

$$\text{and } Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1} \qquad \ldots④$$

iv. $$Z\{a^n \cos n\theta\} = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}, \quad Z\{a^n \sin n\theta\} = \frac{az\sin\theta}{z^2-2z\cos\theta+a^2}$$

By Damping rule, replacing z by $\frac{z}{a}$ in ③and ④, we get

$$Z\{a^n \cos n\theta\} = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2} \text{ and } Z\{a^n \sin n\theta\} = \frac{az\sin\theta}{z^2-2z\cos\theta+a^2}$$

Right Shifting Property

For $n \geq k$, $Z\{u_{n-k}\} = z^{-k}Z\{u_n\}$, k is positive integer

Proof: $Z\{u_{n-k}\} = \sum_{n=0}^{\infty} u_{n-k} z^{-n}$

$$= u_{-k}z^0 + u_{1-k}z^{-1} + \cdots + u_{-1}z^{-k+1} + u_0 z^{-k} + u_1 z^{-(k+1)} + u_2 z^{-(k+2)} + \cdots$$

$$= 0 + u_0 z^{-k} + u_1 z^{-(k+1)} + u_2 z^{-(k+2)} + \cdots \qquad \because u_n = 0 \text{ for } n < 0$$

$$= \sum_{n-k=0}^{\infty} u_{n-k} z^{-n}$$

$$= \sum_{m=0}^{\infty} u_m z^{-k-m}$$

$$= z^{-k} \sum_{m=0}^{\infty} u_m z^{-m}$$

$$= z^{-k} \sum_{n=0}^{\infty} u_n z^{-n}$$

$$= z^{-k} Z\{u_n\}$$

Left Shifting Property

If k is a positive integer $Z\{u_{n+k}\} = z^k\left[Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots - \frac{u_{k-1}}{z^{k-1}}\right]$

Proof: $Z\{u_{n+k}\} = \sum_{n=0}^{\infty} u_{n+k} z^{-n}$

$$= z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$$

$$= z^k\left[u_k z^{-k} + u_{1+k} z^{-(1+k)} + u_{2+k} z^{-(2+k)} + \cdots\right]$$

$$= z^k\left[u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)} + u_k z^{-k} + \cdots\right]$$

$$-z^k\left[u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)}\right]$$

$$= z^k\left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n}\right]$$

$$= z^k\left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n}\right]$$

$$= z^k\left[Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots - \frac{u_{k-1}}{z^{k-1}}\right]$$

In particular for $k = 1,2,3$

$$Z\{u_{n+1}\} = z[Z\{u_n\} - u_0]$$

$$Z\{u_{n+2}\} = z^2\left[Z\{u_n\} - u_0 - \frac{u_1}{z}\right]$$

$$Z\{u_{n+3}\} = z^3\left[Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

Initial Value theorem:

If $Z\{u_n\} = U(z)$, then $u_0 = \lim_{z\to\infty} U(z)$

$$u_1 = \lim_{z\to\infty} z[U(z) - u_0]$$

$$u_2 = \lim_{z\to\infty} z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$$

$$\vdots$$

Proof: By definition $U(z) = Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$

$$\Rightarrow U(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots \qquad \ldots (5)$$

$$\therefore u_0 = \lim_{z\to\infty} U(z) = \lim_{z\to\infty}\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots\right]$$

$$= u_0 + 0 + 0 + 0 + \cdots = u_0$$

Again from (5), we get

$$U(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots$$

$$\Rightarrow z[U(z) - u_0] = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \cdots$$

$$\Rightarrow \lim_{z\to\infty} z[U(z) - u_0] = \lim_{z\to\infty}\left[u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \cdots\right] = u_1$$

Similarly $u_2 = \lim_{z\to\infty} z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$

Note: Initial value theorem may be used to determine the sequence u_n from the given function $U(z)$

Final Value theorem:

If $Z\{u_n\} = U(z)$, then $\lim_{n\to\infty} u_n = \lim_{z\to 1}(z-1)U(z)$

Proof: $Z\{u_{n+1} - u_n\} = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$

$$\Rightarrow Z\{u_{n+1}\} - Z\{u_n\} = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

$$\Rightarrow z[Z\{u_n\} - u_0] - Z\{u_n\} = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

By using left shifting property for $k = 1$

$$\Rightarrow (z-1)Z\{u_n\} - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $(z-1)U(z) - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n} \quad \because Z\{u_n\} = U(z)$

Taking limits $z \to 1$ on both sides

$$\lim_{z\to 1}(z-1)U(z) - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n)$$

or $\lim_{z\to 1}(z-1)U(z) - u_0 = \lim_{n\to\infty}[(u_1 - u_0) + (u_2 - u_1) + \cdots + (u_{n+1} - u_n)]$

$$= \lim_{n\to\infty}[u_{n+1}] - u_0$$

$$\Rightarrow \lim_{z\to 1}(z-1)U(z) = u_\infty$$

or $\lim_{z\to 1}(z-1)U(z) = \lim_{n\to\infty} u_n$

Note: Initial value and final value theorems determine the value of u_n for $n = 0$ and for $n \longrightarrow \infty$ from the given function $U(z)$.

Convolution theorem

Convolution of two sequences u_n and v_n is defined as $u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$

Convolution theorem for Z-transforms states that

If $U(z) = Z\{u_n\}$ and $V(z) = Z\{v_n\}$, then $Z\{u_n * v_n\} = U(z).V(z)$

Proof: $U(z).V(z) = Z\{u_n\}.Z\{v_n\}$

$$= [\sum_{n=0}^{\infty} u_n z^{-n}].[\sum_{n=0}^{\infty} v_n z^{-n}]$$

$$= \left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots + \frac{u_n}{z^n} + \cdots\right].\left[v_0 + \frac{v_1}{z} + \frac{v_2}{z^2} + \cdots + \frac{v_n}{z^n} + \cdots\right]$$

$$= (u_0 v_0) + (u_0 v_1 + u_1 v_0) z^{-1} + (u_0 v_2 + u_1 v_1 + u_2 v_0) z^{-2} + \cdots$$

$$= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \cdots + u_n v_0) z^{-n}$$

$$= \sum_{n=0}^{\infty} (\sum_{m=0}^{n} u_m v_{n-m}) z^{-n}$$

$\Rightarrow U(z).V(z) = Z\{\sum_{m=0}^{n} u_m v_{n-m}\}$ $\qquad \because \sum_{n=0}^{\infty} u_n z^{-n} = Z\{u_n\}$

$\Rightarrow U(z).V(z) = Z\{u_n * v_n\}$ $\qquad \because u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$