2.3 Exact First Order Differential Equation

$M(x,y)\cdot dx + N(x,y)\cdot dy = 0$

تكون المعادلة التفاضلية تامة (Exact) إذا توفر الشرط التالي:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If an equation can be written to the form:

$M(x,y)\cdot dx + N(x,y)\cdot dy = 0$

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$M(x,y) = \frac{\partial f}{\partial x}, \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$N(x,y) = \frac{\partial f}{\partial y}, \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

and have the property that:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ is said to be exact equation.

To find $f(x,y)$:

$$M(x,y) = \frac{df}{dx} \;\rightarrow\; \int df = \int M(x,y)\cdot dx$$

$$\therefore f(x,y) = \int M(x,y)\cdot dx + g(y)$$

$$\frac{df}{dy} = \int \frac{\partial M}{\partial y} \cdot dx + \frac{dg}{dy} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (1)$$

$$\because N = \frac{df}{dy} \dots \dots \dots \dots \dots \dots (2)$$

Sub eq. (1) and eq. (2) to get:

$$\int \frac{\partial M}{\partial y} \cdot dx + \frac{dg}{dy} = N \quad \rightarrow$$

$$\frac{dg}{dy} = N - \int \frac{\partial M}{\partial y} \cdot dx$$

Then integral ∂g to find g(y) to get $f(x, y)$.

Example (1): Show that $y \cdot dx + x \cdot dy = 0$ is exact equation?

Solve:

$$M = y, \quad N = x$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1 \end{aligned}\right\} \rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$$

$$\therefore Exact\ function$$

Example (2): Show that the equation $(x^2 + y^2) \cdot dx + (2xy + \cos y) \cdot dy = 0$ is exact equation and then find $f(x, y)$?

Solve:

$$M = (x^2 + y^2), \quad N = (2xy + \cos y)$$

$$\left.\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2y\right\} \quad \rightarrow \quad \because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y$$

$$\therefore Exact\ function$$

To find $f(x, y)$:

$$M = \frac{df}{dx} = x^2 + y^2 \quad \rightarrow \quad \int df = \int (x^2 + y^2) \cdot dx$$

$$\therefore f(x, y) = \frac{x^3}{3} + xy^2 + g(y)$$

$$\frac{df}{dy} = 2xy + \frac{dg}{dy} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (1)$$

$$\because N = \frac{df}{dy} = 2xy + \cos y \dots \dots \dots \dots \dots \dots (2)$$

Sub eq. (1) and eq. (2) to get:

$$2xy + \frac{dg}{dy} = 2xy + \cos y \quad \rightarrow$$

$$\frac{dg}{dy} = \cos y \quad \rightarrow \quad \int dg = \int \cos y \cdot dy + c$$

$$g(y) = \sin y + c$$

$$\therefore f(x, y) = \frac{x^3}{3} + xy^2 + \sin y + c$$

Example (3): Solve $\boldsymbol{\frac{dy}{dx} = \frac{xy^2 - 1}{1 - x^2 y}}$, **if** $\boldsymbol{f(x, y) = 0 \; at \; x = 0, y = 1}$?

Solve:

$$(xy^2 - 1)dx = (1 - x^2 y)dy \quad \rightarrow \quad (xy^2 - 1)dx - (1 - x^2 y)dy = 0$$

$$(xy^2 - 1)dx + (x^2 y - 1)dy = 0$$

$$M = (xy^2 - 1), \quad N = (x^2 y - 1)$$

$$\left.\frac{\partial M}{\partial y} = 2xy, \quad \frac{\partial N}{\partial x} = 2xy\right\} \quad \rightarrow \quad \because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2xy$$

$$\therefore Exact\ function$$

To find $f(x, y)$:

$$M = \frac{df}{dx} = (xy^2 - 1) \quad \rightarrow \quad \int df = \int (xy^2 - 1) \cdot dx$$

$$\therefore f(x, y) = \frac{x^2y^2}{2} - x + g(y)$$

$$\frac{df}{dy} = x^2y + \frac{dg}{dy} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\because N = \frac{df}{dy} = x^2y - 1 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Sub eq. (1) and eq. (2) to get:

$$x^2y + \frac{dg}{dy} = x^2y - 1 \quad \rightarrow \quad \frac{dg}{dy} = -1$$

$$\int dg = \int -1 \cdot dy + c$$

$$g(y) = -y + c$$

$$\therefore f(x, y) = \frac{x^2y^2}{2} - x - y + c$$

$$\because f(x, y) = 0 \ \ at \ \ x = 0, y = 1$$

$$0 = \frac{0 \times 1}{2} - 0 - 1 + c \quad \rightarrow \quad c = 1$$

$$\therefore f(x, y) = \frac{x^2y^2}{2} - x - y + 1$$

Problems:

1- Solve the differential equation $(x + y^2)\,dx + (2xy + 1)\,dy = 0$ and find particular solution if the function $f(x, y) = 0$ at (0,2)?

Answer: $f(x, y) = \frac{1}{2}x^2 + xy^2 + y - 2$

2- Find value of (a) that makes the equation $(3x^2 + y^2) \cdot dx + axy \cdot dy = 0$ exact?

Answer: $a = 2$