Inspection (Direct inversion) method

Sometimes by observing the coefficients in the given series $U(z)$, it is possible to find the sequence u_n as illustrated in the given examples.

Example 11 Find u_n if $U(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \cdots$

Solution: Given that $U(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$ ...①

Also by the definition of Z-transform $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$ ...②

.Comparing ①and ②, we get $u_n = \left(\frac{1}{2}\right)^n$

Example12 Find u_n if $U(z) = \frac{z^3}{(z-1)^3}$

Solution: $U(z) = \frac{z^3}{(z-1)^3} = \left(\frac{z-1}{z}\right)^{-3} = \left(1 - \frac{1}{z}\right)^{-3}$

$\therefore U(z) = 1 + \frac{3}{z} + \frac{6}{z^2} + \frac{10}{z^3} + \cdots$

$\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$

$\Rightarrow U(z) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} z^{-n}$

Comparing with $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, we get $u_n = \frac{(n+1)(n+2)}{2}$

Example13 Find inverse Z-transform of $3+\frac{2z}{z-1}-\frac{z}{2z-1}$

Solution: Given that $U(z)=3+\frac{2z}{z-1}-\frac{z}{2z-1}$

$$\therefore u_n = 3Z^{-1}[1] + 2Z^{-1}\left[\frac{z}{z-1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right]$$

$$\Rightarrow u_n = 3\delta(n) + 2u(n) - \frac{1}{2}\left(\frac{1}{2}\right)^n = 3\delta(n) + 2u(n) - \left(\frac{1}{2}\right)^{n+1}$$

$\because Z^{-1}[1]=\delta(n)$, $Z^{-1}\left[\frac{z}{z-1}\right]=u(n)$, $Z^{-1}\left[\frac{z}{z-a}\right]=a^n$ where $\delta(n)$ and $u(n)$ are unit impulse and unit step sequences respectively.

Example14 Find inverse Z-transform of $2z^{-2}-\frac{z^{-3}}{z-1}+\frac{2z^{-5}}{2z-1}$

Solution: Given that $U(z)=2z^{-2}-\frac{z^{-3}}{z-1}+\frac{2z^{-5}}{2z-1}$

$$\therefore u_n = 2Z^{-1}[z^{-2}.1] - Z^{-1}\left[z^{-4}.\frac{z}{z-1}\right] + Z^{-1}\left[z^{-6}\frac{z}{z-\frac{1}{2}}\right]$$

$$\Rightarrow u_n = 2\delta(n-2) - u(n-4) + \left(\frac{1}{2}\right)^{n-6} u(n-6)$$

$\because$ From Right shifting property $Z^{-1}[z^{-k}U(Z)] = u_{n-k}$

Direct division method

Direct division is one of the simplest methods for finding inverse Z -transform and can be used for almost every type of expression given in fractional form.

Example15 Find the inverse Z-transform of $\frac{z}{z^2-3z+2}$

Solution: Given that $U(z) = \frac{z}{z^2-3z+2}$

By actual division, we get

$$
\begin{array}{r|l}
 & z^{-1} + 3z^{-2} + 7z^{-3} \\
\hline
z^2 - 3z + 2 & z \\
 & \underline{z - 3 + 2z^{-1}} \\
 & \quad 3 - 2z^{-1} \\
 & \quad \underline{3 - 9z^{-1} + 6z^{-2}} \\
 & \qquad 7z^{-1} - 6z^{-2} \\
 & \qquad \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \\
 & \qquad\qquad 15z^{-2} - 14z^{-3} \\
 & \qquad\qquad \vdots
\end{array}
$$

$$\Rightarrow U(z) = z^{-1} + 3z^{-2} + 7z^{-3} + \cdots$$

$$= \sum_{n=0}^{\infty} (2^n - 1) z^{-n}$$

$$\therefore\ u_n = 2^n - 1$$

Example16 Find the inverse Z-transform of $\frac{4z^2+2z}{2z^2-3z+1}$

Solution: Given that $U(z)=\frac{4z^2+2z}{2z^2-3z+1}$, by actual division, we get

$$\begin{array}{r|l} & 2+4z^{-1}+5z^{-2} \\ 2z^2-3z+1 & 4z^2+2z \\ & \underline{4z^2-6z+2} \\ & 8z-2 \\ & \underline{8z-12+4z^{-1}} \\ & 10-4z^{-1} \\ & \underline{10-15z^{-1}+5z^{-2}} \\ & 11z^{-1}-5z^{-2} \\ & \vdots \end{array}$$

$$\Rightarrow U(z)=2+4z^{-1}+5z^{-2}+\cdots$$

$$=\sum_{n=0}^{\infty}(6-2^{2-n})z^{-n}$$

$$\therefore\ u_n=6-2^{2-n}$$

$$\therefore\ u_n=6-4\left(\frac{1}{2}\right)^n$$

Partial fractions method

Partial fractions method can be used only if order of expression in the numerator is less than or equal to that in the denominator. If order of expression in the numerator is greater, then the fraction may be brought to desired form by direct division. Partial fractions are formed of the expression $\frac{U(z)}{z}$ as demonstrated in the examples below.

Example17 Find the inverse Z -transform of $\frac{z}{6z^2-5z+1}$

Solution: Given that $U(z) = \frac{z}{6z^2-5z+1}$

$$\therefore \frac{U(z)}{z} = \frac{1}{6z^2-5z+1} = \frac{\text{i.e. } u_n = 2^{-n} - 3^{-n}}{(2z-1)(3z-1)}$$

$$\Rightarrow \frac{U(z)}{z} = \frac{2}{2z-1} - \frac{3}{3z-1}$$

$$\Rightarrow U(z) = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{3}}$$

$$\therefore u_n = \left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n \quad \because Z\{a^n\} = \frac{z}{z-a} \text{ or } Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

i.e. $u_n = 2^{-n} - 3^{-n}$

Example18 Find the inverse Z -transform of $\frac{4z^2+2z}{2z^2-3z+1}$

Solution: Given that $U(z) = \frac{4z^2-2z}{2z^2-3z+1} = \frac{2z(2z-1)}{(2z-1)(z-1)}$

$$\therefore \frac{U(z)}{z} = \frac{2(2z+1)}{(2z-1)(z-1)}$$

By partial fractions, we get

$$\frac{U(z)}{z} = \frac{-8}{2z-1} + \frac{6}{z-1}$$

$$\Rightarrow U(z) = \frac{-8z}{2z-1} + \frac{6z}{z-1}$$

$$\therefore\ u_n = -4Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] + 6Z^{-1}\left[\frac{z}{z-1}\right]$$

$$\Rightarrow\ u_n = -4\left(\frac{1}{2}\right)^n + 6(1)^n \quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

i.e. $u_n = -4\left(\frac{1}{2}\right)^n + 6$

Example19 Find the inverse Z -transform of $\frac{1}{(1-z^{-1})(2-z^{-1})}$

Solution: Given that $U(z) = \frac{1}{(1-z^{-1})(2-z^{-1})}$

Multiplying and dividing by z^2, we get

$$U(z) = \frac{z^2}{z(1-z^{-1})z(2-z^{-1})} = \frac{z^2}{(z-1)(2z-1)}$$

$$\therefore \frac{U(z)}{z} = \frac{z}{(z-1)(2z-1)}$$

By partial fractions, we get

$$\frac{U(z)}{z} = \frac{1}{(z-1)} - \frac{1}{(2z-1)}$$

$$\Rightarrow U(z) = \frac{z}{(z-1)} - \frac{z}{(2z-1)}$$

$$\therefore u_n = Z^{-1}\left[\frac{z}{(z-1)}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right]$$

$$\Rightarrow u_n = (1)^n - \frac{1}{2}\left(\frac{1}{2}\right)^n \quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

i.e. $u_n = 1 - \left(\frac{1}{2}\right)^{n+1}$

Example20 Find the inverse Z -transform of $\frac{4z^2-2z}{z^3-5z^2+8z-4}$

Solution: Given that $U(z) = \frac{4z^2-2z}{z^3-5z^2+8z-4} = \frac{2z(2z-1)}{(z-1)(z-2)^2}$

$$\therefore \frac{U(z)}{z} = \frac{2(2z-1)}{(z-1)(z-2)^2}$$

By partial fractions, we get

$$\frac{U(z)}{z} = \frac{2}{z-1} - \frac{2}{z-2} + \frac{6}{(z-2)^2}$$

$$\Rightarrow U(z) = \frac{2z}{z-1} - \frac{2z}{z-2} + \frac{6z}{(z-2)^2}$$

$$\therefore u_n = 2Z^{-1}\left[\frac{z}{z-1}\right] - 2Z^{-1}\left[\frac{z}{z-2}\right] + 3Z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$\Rightarrow u_n = 2(1)^n - 2(2)^n + 3n(2)^n \quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n \text{ and } Z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n$$

i.e. $u_n = 2 - 2^{n+1} + 3n.2^n$

Method of residues (Inverse integral)

By using the theory of complex variables, it can be shown that the inverse Z-transform is given by $u_n = \frac{1}{2\pi i}\oint_c U(z)z^{n-1}dz$ = sum of residues of $U(z)$

where c is the closed contour which contains all the insolated singularities of $U(z)$in the region of convergence.

Method of residues is one of the most efficient methods and can be used to find the inverse Z- transform where partial fractions are tedious to find.

Example21 Find the inverse z-transform of $\frac{z}{z^2+7z+10}$

Solution: $U(z) = \frac{z}{z^2+7z+10}$

Now $u_n = \frac{1}{2\pi i}\oint_c U(z)z^{n-1}dz$

$$\Rightarrow u_n = \frac{1}{2\pi i}\oint_c \frac{z}{z^2+7z+10}z^{n-1}dz$$

$$= \frac{1}{2\pi i}\oint_c \frac{z^n}{z^2+7z+10}dz$$

$$= \frac{1}{2\pi i}\oint_c \frac{z^n}{(z+2)(z+5)}dz$$

There are two simple poles at $z = -2$ and $z = -5$

Residue at $z = -2$ is given by $\lim_{z\to-2}(z+2)\frac{z^n}{(z+2)(z+5)} = \frac{(-2)^n}{3}$

Residue at $z = -5$ is given by $\lim_{z\to-5}(z+5)\frac{z^n}{(z+2)(z+5)} = \frac{(-5)^n}{-3}$

$\therefore u_n$= sum of residues $= \frac{(-2)^n}{3} + \frac{(-5)^n}{-3} = \frac{1}{3}\left\{(-2)^n - (-5)^n\right\}$

Example22 Find the inverse z-transform of $\frac{z^2+z}{(z-1)(z^2+1)}$

Solution: $U(z) = \frac{z^2+z}{(z-1)(z^2+1)}$

Now $u_n = \frac{1}{2\pi i}\oint_c U(z)z^{n-1}dz$

$$\Rightarrow u_n = \frac{1}{2\pi i}\oint_c \frac{z^2+z}{(z-1)(z^2+1)}z^{n-1}dz$$

$$= \frac{1}{2\pi i}\oint_c \frac{z^n(z+1)}{(z-1)(z+i)(z-i)}dz$$

There are three simple poles at $z = 1$, $z = -i$ and $z = i$

Residue at $z = 1$ is given by $\lim_{z\to 1}(z-1)\frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = 1$

Residue at $z = -i$ is given by $\lim_{z\to -i}(z+i)\frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = -\frac{1}{2}(-i)^n$

Residue at $z = i$ is given by $\lim_{z\to i}(z-i)\frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = -\frac{1}{2}i^n$

$\therefore u_n$ = sum of residues $= 1 - \frac{1}{2}(-i)^n - \frac{1}{2}i^n = 1 - \frac{1}{2}\{(-i)^n + i^n\}$

Example23 Find the inverse z-transform of $\frac{z(z+1)}{(z-1)^3}$

Solution: $U(z) = \frac{z(z+1)}{(z-1)^3}$

Now $u_n = \frac{1}{2\pi i}\oint_c U(z)z^{n-1}dz$

$\Rightarrow u_n = \frac{1}{2\pi i}\oint_c \frac{z^n(z+1)}{(z-1)^3}dz$

Here $z = 1$ is a pole of order 3

Residue at $z = 1$ is given by $\frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left[\frac{(z-1)^3 z^n(z+1)}{(z-1)^3}\right]$

$$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\,[z^n(z+1)]$$

$$= \frac{1}{2!}\lim_{z\to 1}\frac{d}{dz}\,[(n+1)z^n + nz^{n-1}]$$

$$= \frac{1}{2!}\lim_{z\to 1}[(n+1)nz^{n-1} + n(n-1)z^{n-2}]$$

$$= [n^2 + n + n^2 - n] = n^2$$

$\therefore u_n = \text{Sum of residues} = n^2$

Power series method

In this method, we find the inverse Z - transform by expanding $U(z)$ in power series.

Example 24 Find u_n if $U(z) = \log\frac{z}{z+1}$

Solution: Given $U(z) = \log\frac{z}{z+1} = \log\left(\frac{z+1}{z}\right)^{-1} = -\log\frac{z+1}{z} = -\log\left(1+\frac{1}{z}\right)$

$$\therefore U(z) = -\log(1+y) \qquad \text{Putting } \frac{1}{z} = y$$

$$= -y + \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4} - \cdots$$

$$\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\Rightarrow U(z) = -\frac{1}{z} + \frac{1}{2z^2} - \frac{1}{3z^3} + \frac{1}{4z^4} - \cdots \qquad \because y = \frac{1}{z}$$

$$\Rightarrow U(z) = 0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n}$$

Comparing with $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, we get

$$u_n = \begin{cases} 0 & \text{for } n = 0 \\ \frac{(-1)^n}{n}, & otherwise \end{cases}$$

Convolution theorem method

Convolution theorem for Z-transforms states that:

If $U(z) = Z\{u_n\}$ and $V(z) = Z\{v_n\}$, then $Z\{u_n * v_n\} = U(z).V(z)$

$$\Rightarrow Z^{-1}[U(z).V(z)] = u_n * v_n$$

Example25 Find the inverse z-transform of $\frac{z^2}{(z-1)(2z-1)}$ using convolution theorem.

Solution: Let $U(z) = Z\{u_n\} = \frac{z}{(z-1)}$ and $V(z) = Z\{v_n\} = \frac{z}{(2z-1)} = \frac{1}{2}\left(\frac{z}{z-\frac{1}{2}}\right)$

Clearly $u_n = (1)^n$ and $u_n = \frac{1}{2}\left(\frac{1}{2}\right)^n$ $\because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

Now by convolution theorem $Z^{-1}[U(z).V(z)] = u_n * v_n$

$$\Rightarrow Z^{-1}\left[\frac{z^2}{(z-1)(2z-1)}\right] = (1)^n * \left(\frac{1}{2}\right)^{n+1}$$

We know that $u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$

$$= \sum_{m=0}^{n} (1)^m \left(\frac{1}{2}\right)^{n+1-m}$$

$$= \left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{2}\right)^{n} + \left(\frac{1}{2}\right)^{n-1} + \cdots + \frac{1}{2}$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}}\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)\right]$$

$$\because S_n = \frac{a}{1-r}(1 - r^n)$$

$$= \frac{1}{2}\left[2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)\right]$$

$$= 1 - \left(\frac{1}{2}\right)^{n+1}$$

Exercise 4A

1. Find the Z–transform of $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$
2. Find the Z–transform of $u_n = \frac{1}{(n-p)!}$
3. Find the inverse Z–transform of $u_n = \frac{2z}{(z-1)(z^2+1)}$
4. Solve the difference equation $y_{x+2} + 4y_{x+1} + 3y_x = 3^x$, $y_0 = 0, y_1 = 1$ using Z–transforms

Answers

1. $\frac{2z}{z^2-8z+15}$, $3 < |z| < 5$
2. $z^{-p} e^{\frac{1}{z}}$
3. $1 - \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}$
4. $y_x = \frac{1}{24} 3^x - \frac{5}{12}(-3)^x + \frac{3}{8}(-1)^x$