Introduction to Numerical Analysis

1.1 Analysis versus Numerical Analysis

The word ***analysis*** in mathematics usually means who to solve a problem through equations. The solving procedures may include algebra, calculus, differential equations, or the like.

Numerical analysis is similar in that problems solved, but the only procedures that are used are arithmetic: add, subtract, multiply, divide and compare.

Differences between ***analytical solutions*** and ***numerical solutions***:

1) An analytical solution is usually given in terms of mathematical functions. The behavior and properties of the function are often apparent. However, a numerical solution is always an approximation. It can be plotted to show some of the behavior of the solution.

2) An analytical solution is not always meaningful by itself.

   Example: $\sqrt{3}$ as one of the roots of $x^3 - x^2 - 3x + 3 = 0$.

3) While the numerical solution is an approximation, it can usually be evaluated as accurate as we need. Actually, evaluating an analytic solution numerically is subject to the same errors.

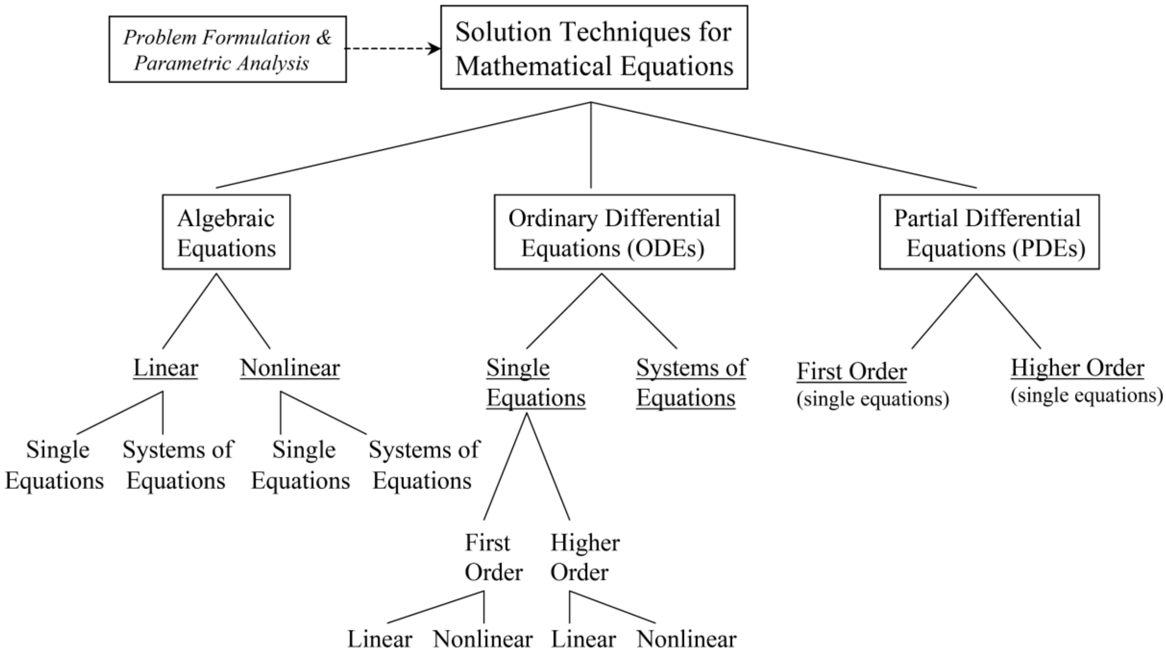
1.2 Computers and Numerical Analysis

Numerical Methods＋Programs $\xrightarrow{\text{Computers}}$ Numerical Analysis

- As you will learn enough about many ***numerical methods***, you will be able to write ***programs*** to implement them.
- Programs can be written in any computer language. In this course all programs will be written in Matlab environment.
- Actually, writing programs is not always necessary. Numerical analysis is so important that extensive commercial software packages are available.

1.3 Types of Equations

The equations is divided into three main categories such as in below figure:-



1.4 Kinds of Errors in Numerical Procedures

The total error comprises of:

1) ***Model Error***: due to the mismatch between the physical situation and the mathematical model.

2) ***Data Error***: due to the measurements of doubtful accuracy.

3) ***Human Error***: due to human blunders.

4) ***Propagated Error***: the error in the succeeding steps of a process due to an occurrence of an earlier error.

5) ***Truncation Error***: the notion of truncation error usually refers to errors introduced when a more complicated mathematical expression is “replaced” with a more elementary formula. This formula itself may only be approximated to the true values, thus would not produce exact answers.

Example 1.1:

Truncation of an infinite series to a finite series to a finite number of terms leads to the truncation error. For example, the Taylor series of exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

If only four terms of the series are used, then

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^1 \approx 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} \quad = 2.66667$$

The truncation error would be the unused terms of the Taylor series, which then are

$$E_t = \frac{x^4}{4!} + \frac{x^5}{5!} + \Lambda \;= \frac{1^4}{4!} + \frac{1^5}{5!} + \Lambda \qquad \cong 0.0516152$$

Check a few Taylor series approximations of the number ex, for x = 1, n = 2, 3 and 4. Given that e1 = 2.718281.

Order of n	Approximation for ex	Absolute error	Percent relative error
2	2.500000	0.218281	8.030111%
3	2.666667	0.051614	1.898774%
4	2.708333	0.00995	0.365967%

6) ***Round-Off Error***: A **round-off error**, also called **rounding error**, is the difference between the calculated approximation of a number and its exact mathematical value **due** to rounding

Example 1.2:

Numbers such as π, e, or $\sqrt{3}$ cannot be expressed by a fixed number of decimal places. Therefore they cannot be represented exactly by the computer.

Consider the number π. It is irrational, i.e. it has infinitely many digits after the period: π = 3.1415926535897932384626433832795.....

The round-off error computer representation of the number π depends on how many digits are left out.

Let the true value for π is 3.141593.

Number of digits (Decimal digit)	Approximation for π	Absolute error	Percent relative error
1	3.1	0.041593	1.3239%
2	3.14	0.001593	0.0507%
3	3.142	0.000407	0.0130%

1.5 Errors in Numerical Procedures

There are two common ways to express the size of the error in a computed result: ***absolute error*** and ***relative error***.

- Absolute error = | true value – approximate value |, which is usually used when the magnitude of the true value is small.

- Relative error = $\dfrac{|\text{true value - approximate value}|}{|\text{true value}|}$, which is a desirable one.

While

$$\text{Percent relative error}, \varepsilon_t = \left|\frac{\text{true value} - \text{approximate value}}{\text{true value}}\right| \times 100\%$$