$$C(x,t) = \frac{4(C_0 - C_i)}{\pi} \sum_{K=0}^{\infty} e^{-\left(\frac{(2K+1)\pi\sqrt{D}}{L}\right)^2 \cdot t} \cdot \frac{1}{(2K+1)} \sin \frac{(2K+1)\pi}{L} x + C_i$$

Ex: Solve the partial differential equation,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

for the conditions,

i) $\theta(0,t) = 20$

ii) $\theta(20,t) = 20$

iii) $$\theta(x,0) = \begin{cases} 120 & 0 \leqslant x \leqslant 15 \\ 30 & 15 \leqslant x \leqslant 20 \end{cases}$$

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$\bar{\theta} = \theta - 20$, $\bar{\theta}(x,t) = \theta(x,t) - 20$

i) $\bar{\theta}(0,t) = 20 - 20 = 0$

ii) $\bar{\theta}(20,t) = 20 - 20 = 0$

iii) $\bar{\theta}(x,0) = \begin{cases} 120 - 20 = 100 & 0 \leq x \leq 15 \\ 30 - 20 = 10 & 15 \leq x \leq 20 \end{cases}$

$$\frac{\partial \bar{\theta}}{\partial t} = h^2 \frac{\partial^2 \bar{\theta}}{\partial x^2}$$

The general solution

$$\bar{\theta}(x,t) = e^{-\beta^2 t} \left(A \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x \right)$$

B.C. 1 $\quad x = 0 \quad \bar{\theta} = 0$

$$0 = e^{-\beta^2 \cdot t} \left(A(1) + B(0) \right) \Rightarrow 0 = e^{-\beta^2 \cdot t} A$$

$$e^{-\beta^2 \cdot t} \neq 0 \Rightarrow A = 0$$

$$\bar{\theta}(x,t) = e^{-\beta^2 \cdot t} \left(B . \sin \frac{\beta}{h} x \right)$$

B.C. 2 : $x = 20$ $\bar{\theta} = 0$

$$0 = e^{-\beta^2 \cdot t} . B . \sin \frac{\beta}{h} (20)$$

$$e^{-\beta^2 \cdot t} \neq 0 , B \neq 0 \Rightarrow \sin \frac{\beta}{h} (20) = 0$$

$$\frac{\beta}{h} 20 = n\pi \Rightarrow \beta = \frac{n\pi h}{20}$$

$$\bar{\theta}(x,t) = B \sin \frac{n\pi}{20} x . e^{-\left(\frac{n\pi h}{20}\right)^2 \cdot t}$$

More general

$$\bar{\theta}(x,t) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{20} x . e^{-\left(\frac{n\pi h}{20}\right)^2 \cdot t}$$

I.C. : $t = 0$ $\bar{\theta} = 100$ $0 \leq x \leq 15$

& $\bar{\theta} = 10$ $15 \leq x \leq 20$

$$\bar{\theta}(x,0) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{20} x . 1$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin nx \, dx$$

$$B_n = \frac{2}{20} \left[\int_0^{15} 100 . \sin \frac{n\pi}{20} x \, dx + \int_{15}^{20} 10 . \sin \frac{n\pi}{20} x \, dx \right]$$

$$B_n = \frac{2}{20}\left[\frac{20\times 100}{n\pi}\left(-\cos\frac{n\pi}{20}x\right)_0^{15} + \frac{20\times 10}{n\pi}\left(-\cos\frac{n\pi}{20}x\right)_{15}^{20}\right]$$

$$B_n = \frac{200}{n\pi}\left(1-\cos\frac{3}{4}n\pi\right) + \frac{20}{n\pi}\left(\cos\frac{3}{4}n\pi - \cos n\pi\right)$$

$$\cos n\pi = (-1)^n$$

$$B_n = \frac{20}{n\pi}\left(10 - 10\cos\frac{3}{4}n\pi + \cos\frac{3}{4}n\pi - \cos n\pi\right)$$

$$B_n = \frac{20}{n\pi}\left(10 - 9\cos\frac{3}{4}n\pi - (-1)^n\right)$$

$$\bar{\theta}(x,t) = \sum_{n=0}^{\infty}\frac{20}{n\pi}\left(10 - 9\cos\frac{3}{4}n\pi - (-1)^n\right)\cdot \sin\frac{n\pi}{20}x \cdot e^{-\left(\frac{n\pi h}{20}\right)^2\cdot t}$$

$$\bar{\theta}(x,t) = \theta(x,t) - 20 \Rightarrow \theta(x,t) = \bar{\theta}(x,t) + 20$$

$$\therefore \theta(x,t) = 20 + \sum_{n=0}^{\infty}\frac{20}{n\pi}\left(10 - 9\cos\frac{3n\pi}{4} - (-1)^n\right)\sin\frac{n\pi}{20}x \cdot e^{-\left(\frac{n\pi h}{20}\right)^2 t}$$

Ex: Solve the partial differential equation

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2} \quad , \quad h = 1$$

For the following condition

i) $t = 0 \quad \theta = 0 \qquad \theta(x,0) = 0$

ii) $x = 0 \quad \theta = 60 \qquad \theta(0,t) = 60$

iii) $x = 100 \quad \frac{\partial \theta}{\partial x} = 0 \qquad \frac{\partial \theta}{\partial x}(100,t) = 0$

$$\bar{\theta}(x,t) = \theta(x,t) - 60$$

$$\frac{\partial \bar{\theta}}{\partial x} = \frac{\partial \theta}{\partial x} - 0 \Rightarrow \frac{\partial \bar{\theta}}{\partial x} = \frac{\partial \theta}{\partial x}$$

i) $t = 0 \quad \bar{\theta} = 0 - 60 = -60 \qquad \bar{\theta}(x,0) = -60$

ii) $x = 0 \quad \bar{\theta} = 60 - 60 = 0 \qquad \bar{\theta}(0,t) = 0$

iii) $x = 100 \quad \frac{\partial \bar{\theta}}{\partial x} = 0 \qquad \frac{\partial \bar{\theta}}{\partial x}(100,t) = 0$

$$\frac{\partial \bar{\theta}}{\partial t} = h^2 \frac{\partial^2 \bar{\theta}}{\partial x^2} \Rightarrow \frac{\partial \bar{\theta}}{\partial t} = \frac{\partial^2 \bar{\theta}}{\partial x^2} \quad , \quad h = 1$$

$$\bar{\theta}(x,t) = e^{-\beta^2 t}(A \cos \beta x + B \sin \beta x) \quad , \quad h = 1$$

B.C. 1 $\quad x = 0 \quad \bar{\theta} = 0$

$$\bar{\theta}(0,t) = e^{-\beta^2 t}(A \cos(0) + B \sin(0)) = 0$$

$$\bar{\theta}(0,t) = e^{-\beta^2 t} \cdot A = 0$$

$$e^{-\beta^2 t} \neq 0 \Rightarrow A = 0$$

$$\bar{\theta}(x,t) = e^{-\beta^2 t}(B \sin \beta x)$$

B. C. 2 $\quad x = 100 \quad \frac{\partial \bar{\theta}}{\partial x} = 0$

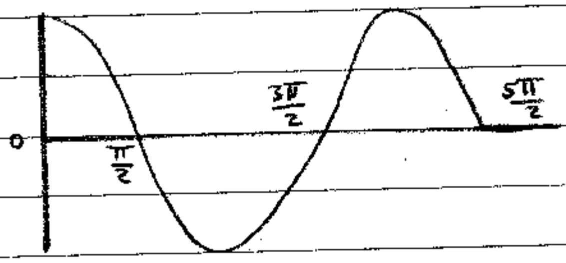
$$\frac{\partial \bar{\theta}}{\partial x} = e^{-\beta^2 t}(B . \beta . \cos \beta x)$$

$$\frac{\partial \bar{\theta}}{\partial x}(100,t) = e^{-\beta^2 t} . B . \beta . \cos \beta(100) = 0$$

$$B \neq 0 \ , \ e^{-\beta^2 t} \neq 0 \Rightarrow \cos \beta(100) = 0$$

$$100 \beta = n \frac{\pi}{2}$$

$$\beta = \frac{n}{200} \pi$$



$$\bar{\theta}(x,t) = e^{-(\frac{n}{200}\pi)^2 t} . B . \sin \frac{n}{200} \pi x$$

More general

$$\bar{\theta}(x,t) = \sum_{n=0}^{\infty} e^{-(\frac{n}{200}\pi)^2 . t} . B_n . \sin \frac{n}{200} \pi x$$

I. C. $\quad t = 0 \quad \bar{\theta} = -60$

$$\bar{\theta}(x,0) = \sum_{n=0}^{\infty} 1 . B_n . \sin \frac{n}{200} \pi x = -60$$

This is a half Fourier series :

$$B_n = \frac{2}{L} \int_0^L f(x) \sin nx \, dx$$

$$B_n = \frac{2}{100} \int_0^{100} (-60) \sin \frac{n}{200} \pi x \, dx$$

$$B_n = \frac{-2(60)}{100} \cdot \frac{200}{n\pi} \left[-\cos \frac{n\pi}{200} x \right]_0^{100}$$

$$B_n = \frac{240}{n\pi} \left[\cos \frac{n\pi}{2} - 1 \right]$$

put $n = 2K+1$, $K = 0, 1, 2, 3, \ldots$

$$B_K = \frac{240}{(2K+1)\pi} \left[\cos \frac{(2K+1)\pi}{2} - 1 \right]$$

$$\cos \frac{(2K+1)\pi}{2} = 0$$

$$\therefore \; B_K = \frac{-240}{(2K+1)\pi}$$

$$\bar{\theta}(x,t) = \sum_{n=0}^{\infty} e^{-\left(\frac{(2K+1)\pi}{200}\right)^2 \cdot t} \cdot \frac{-240}{(2K+1)\pi} \cdot \sin \frac{(2K+1)\pi}{200} x$$

$$\bar{\theta}(x,t) = \theta(x,t) - 60 \Rightarrow \theta(x,t) = \bar{\theta}(x,t) + 60$$

$$\theta(x,t) = 60 + \sum_{n=0}^{\infty} e^{-\left(\frac{(2K+1)\pi}{200}\right)^2 \cdot t} \cdot \frac{-240}{(2K+1)\pi} \cdot \sin \frac{(2K+1)\pi}{200} x$$

Ex: Solve the partial differential equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

For the following condition

i) $t = 0$ $\quad T = T_0$
ii) $x = 0$ $\quad T = T_0$
iii) $x = L$ $\quad T = T_1$

Let $\bar{T} = T - T_s$ (T_s at steady state).

$$\frac{\partial^2 T}{\partial x^2} = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = 0 \Rightarrow \frac{\partial T}{\partial x} = A$$

$$T = Ax + B$$

B.C. 1 $\quad x = 0 \quad T = T_0$

$$T = Ax + B \Rightarrow T_0 = A(0) + B \Rightarrow B = T_0$$

B.C. 2 $\quad x = L \quad T = T_1$

$$T = Ax + B \Rightarrow T_1 = AL + B \Rightarrow T_1 = AL + T_0$$

$$\therefore \quad A = \frac{T_1 - T_0}{L}$$

$$T = Ax + B \Rightarrow T = \frac{T_1 - T_0}{L} x + T_0$$

$$\bar{T} = T - \left[\frac{T_1 - T_0}{L} x + T_0\right]$$

$$\frac{\partial \bar{T}}{\partial t} = \alpha \frac{\partial^2 \bar{T}}{\partial x^2}$$

i) $t = 0 \quad \bar{T} = T_0 - \left[\frac{T_1 - T_0}{L} x + T_0\right]$

$$\bar{T} = \frac{T_0 - T_1}{L} x$$

ii) $x = 0 \quad \bar{T} = T_0 - \left[\frac{T_1 - T_0}{L}(0) + T_0\right] = 0$

iii) $x = L \quad \bar{T} = T_1 - \left[\frac{T_1 - T_0}{L} L + T_0\right] = 0$

For $k = -ve$

$$\bar{T} = e^{-\beta^2 . t} \left(A \cos \frac{\beta}{\sqrt{\alpha}} x + B \sin \frac{\beta}{\sqrt{\alpha}} x \right)$$

B.C. 1 $\quad x = 0 \quad \bar{T} = 0$

$$0 = e^{-\beta^2 . t} \left(A \cos(0) + B \sin(0) \right)$$

$$0 = e^{-\beta^2 . t} . A \Rightarrow e^{-\beta^2 . t} \neq 0 \quad \therefore A = 0$$

$$\bar{T} = e^{-\beta^2 . t} . B \sin \frac{\beta}{\sqrt{\alpha}} x$$

B.C. 2 $\quad x = L \quad \bar{T} = 0$

$$0 = e^{-\beta^2 . t} . B \sin \frac{\beta}{\sqrt{\alpha}} L$$

$$e^{-\beta^2 . t} \neq 0 , \quad B \neq 0 \Rightarrow \sin \frac{\beta}{\sqrt{\alpha}} L = 0$$

$$\frac{\beta}{\sqrt{\alpha}} L = n\pi \quad , \quad n = 0, 1, 2, 3, \ldots$$

$$\beta = \frac{n\pi\sqrt{\alpha}}{L}$$

$$\bar{T} = e^{-\left(\frac{n\pi\sqrt{\alpha}}{L}\right)^2 . t} . B . \sin \frac{n\pi}{L} x$$

More general

$$\bar{T} = \sum_{n=0}^{\infty} e^{-\left(\frac{n\pi\sqrt{\alpha}}{L}\right) . t} . B_n . \sin \frac{n\pi}{L} x$$

I.C. $\quad t=0 \qquad \bar{T} = \frac{T_0 - T_1}{L} x$

$$\frac{T_0 - T_1}{L} x = \sum_{n=0}^{\infty} 1 \cdot B_n \cdot \sin \frac{n\pi}{L} x$$

$$B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin x \, dx$$

$$B_n = \frac{2}{L} \int_0^L \frac{T_0 - T_1}{L} x \cdot \sin \frac{n\pi}{L} x \, dx$$

$$B_n = \frac{2(T_0 - T_1)}{L^2} \int_0^L x \cdot \sin \frac{n\pi}{L} x \, dx$$

$$\int u \, dv = uv - \int v \, du$$

$$\int_0^L x \cdot \sin \frac{n\pi}{L} x \, dx = -x \frac{L}{n\pi} \cos \frac{n\pi}{L} x - \int -\frac{L}{n\pi} \cos \frac{n\pi}{L} x \, dx$$

$$= \frac{-xL}{n\pi} \cos \frac{n\pi}{L} x + \frac{L}{n\pi} \sin \frac{n\pi}{L} x \left(\frac{n\pi}{L} \right) \Big]_0^L$$

$$= \frac{-xL}{n\pi} \cos \frac{n\pi}{L} x + \sin \frac{n\pi}{L} L$$

$$B_n = \frac{2(T_0 - T_1)}{L^2} \left[\overset{0}{\sin n\pi} - \frac{xL}{n\pi} \cos \frac{n\pi}{L} x \right]$$

$$\bar{T} = \sum_{n=0}^{\infty} \frac{-2(T_0 - T_1)}{L^2} \cdot \frac{xL}{n\pi} \cos \frac{n\pi}{L} x \cdot e^{-\left(\frac{n\pi\sqrt{\alpha}}{L}\right)^2 \cdot t} \cdot \sin \frac{n\pi}{L} x$$

$$T = \sum_{n=0}^{\infty} \frac{-2(T_0 - T_1)}{L^2} \cdot \frac{xL}{n\pi} \cos \frac{n\pi}{L} x \cdot e^{-\left(\frac{n\pi\sqrt{\alpha}}{L}\right)^2 \cdot t} \cdot \sin \frac{n\pi}{L} x + \left[\frac{T_1 - T_0}{L} x + T_0 \right]$$

Ex: Find the general solution to the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}$$ Cylinder equation

$u(r,t) = R(r) \cdot T(t)$

$\frac{\partial u}{\partial t} = RT'$

$\frac{\partial u}{\partial r} = TR'$ & $\frac{\partial^2 u}{\partial r^2} = TR''$

$TR'' + \frac{1}{r} TR' = RT'$ $\div RT$

$\frac{R''}{R} + \frac{R'}{rR} = \frac{T'}{T} = K$

for $K < 0$ or $K = -\beta^2$

$$\frac{R'' + \frac{R'}{r}}{R} = -\beta^2 \Rightarrow R'' + \frac{R'}{r} + \beta^2 R = 0$$

$r^2 R'' + rR' + r^2\beta^2 R = 0$

This is Bessel's equation of zero order, the solution for which is,

$$R = A\,J_0(r\beta) + B\,Y_0(r\beta)$$

$\frac{T'}{T} = -\beta^2 \Rightarrow T' + \beta^2 T = 0 \Rightarrow T = C e^{-\beta^2 t}$

The total solution is,

$$u(r,t) = e^{-\beta^2 t}\left(AJ_0(r\beta) + BY_0(r\beta)\right)$$