Lecture: 4
Logic: Part IV

Algebra of Propositions

Propositions satisfy various laws which are listed in the table below. (In this table, T and F are restricted to the truth values "True" and "False," respectively.)

Laws of the algebra of propositions

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv T$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

ARGUMENTS

An *argument* is an assertion that a given set of propositions $P_1, P_2, \ldots, P_n$ called *premises*, yields, another proposition Q, called the *conclusion*. Such an argument is denoted by $P_1, P_2, \ldots, P_n \vdash Q$

The notion of a "logical argument" or "valid argument" is formalized as follows:

Definitions

Valid Argument: An argument $P_1, P_2, \ldots, P_n \vdash Q$ is said to be *valid* if Q is true whenever all the premises $P_1, P_2, \ldots, P_n$ are true.

Fallacy: An argument which is not valid is called *fallacy*.

Examples:

(a) The following argument is valid:

$p, p \rightarrow q \vdash q$ *(Law of Detachment)*

To prove this rule look at the following truth table

p	q	$p \longrightarrow q$	$p \wedge (p \longrightarrow q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

Specifically , p and $p \longrightarrow q$ are true only in case row 1, and in this case q is true .

(b) The following argument is a fallacy:

$p \rightarrow q, q \vdash p$ (prove!).

Example: A fundamental principle of logical reasoning states:
"If p implies q and q implies r, then p implies r"

p	q	r	$[(p$	$\rightarrow$	$q)$	$\wedge$	$(q$	$\rightarrow$	$r)]$	$\rightarrow$	$(p$	$\rightarrow$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	T	F	F	T	F	T	F
F	F	T	F	T	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F
Step			1	2	1	3	1	2	1	4	1	2	1

That is, the following argument is valid: $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$ (***Law of Syllogism***)

This fact is verified by the above truth table which shows that the following proposition is a tautology:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r).$$

Example, consider the following argument:

S_1 : If a man is a bachelor, he is unhappy.
S_2 : If a man is unhappy, he dies young.

Conclusion: *S : Bachelors die young*

Here the statement S below the line denotes the conclusion of the argument, and the statements S_1 and S_2 above the line denote the premises. We claim that the argument $S_1, S_2 \vdash S$ is valid. For the argument is of the form

$$p \rightarrow q,\ q \rightarrow r \vdash p \rightarrow r,$$

where p is "He is a bachelor," q is "He is unhappy" and r is "He dies young".
and by "Law of Syllogism" This argument is valid.

Propositional Functions , Quantifiers

Let A be a given set. *A propositional function* (or an *open sentence* or *condition*) defined on A is an expression $p(x)$, which has the property that $p(a)$ is true or false for each $a \in A$.
The set A is called the *domain* of $p(x)$, and the set Tp of all elements of A for which $p(a)$ is true is called the *truth set* of $p(x)$. In other words,

$$Tp = \{x \mid x \in A, p(x) \text{ is true}\} \text{ or } Tp = \{x \mid p(x)\}$$

Note: Frequently ,when A is some set of numbers, the condition $p(x)$ has the form of an equation or inequality involving the variable x.

Examples:

Find the truth set for each propositional function $p(x)$ defined on the set **N** of positive integers.

(a) Let $p(x)$ be "$x + 2 > 7$." Its truth set is $\{6, 7, 8, \ldots\}$ consisting of all integers greater than 5.
(b) Let $p(x)$ be "$x + 5 < 3$." Its truth set is the empty set ϕ. That is, $p(x)$ is not true for any integer in **N**.
(c) Let $p(x)$ be "$x + 5 > 1$." Its truth set is **N**. That is, $p(x)$ is true for every element in **N**.

Note: In the above examples: If $p(x)$ is propositional function defined on a set A then $p(x)$ could be true for all $x \in A$, for some $x \in A$, or for not $x \in A$.

Remark

Next we discuss quantifiers related to such propositional functions.

Universal Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\forall x \in A)\ p(x) \quad or \quad \forall x\ p(x)$$

which reads "For every x in A, $p(x)$ is a true statement" or, simply, "For all x, $p(x)$."
The symbol ∀, which reads "for all" or "for every" is called the *universal quantifier*.
The above statement is equivalent to the statement

$$T_p = \{x \mid x \in A, p(x)\} = A$$

that is, that the truth set of $p(x)$ is the entire set A. So, we have a conclusion:

If $\{x \mid x \in A, p(x)\} = A$ then $\forall x\ p(x)$ is true; otherwise, $\forall x\ p(x)$ is false.

Examples

(a) The proposition $(\forall n \in \mathbf{N})(n + 4 > 3)$ is true since $\{n \mid n + 4 > 3\} = \{1, 2, 3, \ldots\} = \mathbf{N}$.
(b) The proposition $(\forall n \in \mathbf{N})(n + 2 > 8)$ is false since $\{n \mid n + 2 > 8\} = \{7, 8, \ldots\} \neq \mathrm{N}$.
(c) The symbol ∀ can be used to define the intersection of an indexed collection $\{A_i \mid i \in I\}$ of sets A_i as follows:

$$\cap(A_i \mid i \in I) = \{x \mid \forall i \in I, x \in A_i\}.$$

Existential Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\exists x \in A)\ p(x) \text{ or } \exists x, p(x),$$

which reads "There exists an x in A such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$."
The symbol ∃, which reads "there exists" or "for some" or "for at least one" is called the ***existential quantifier***. Above statement is equivalent to the statement

$$T_p = \{x \mid x \in A, p(x)\} \neq \phi.$$

i.e., that the truth set of $p(x)$ is not empty. Accordingly, $\exists x\ p(x)$, that is, $p(x)$ preceded by the quantifier ∃, does have a truth value. Specifically:

If $\{x \mid p(x)\} \neq \phi$ then $\exists x\ p(x)$ is true; otherwise, $\exists x\ p(x)$ is false.

Examples

(a) The proposition $(\exists n \in \mathbf{N})(n + 4 < 7)$ is true since $\{n \mid n + 4 < 7\} = \{1, 2\} \neq \phi$.

(b) The proposition $(\exists n \in \mathbf{N})(n + 6 < 4)$ is false since $\{n \mid n + 6 < 4\} = \phi$.

(c) The symbol ∃ can be used to define the union of an indexed collection $\{A_i \mid i \in I\}$ of sets Ai as follows:

$$\cup(A_i \mid i \in I) = \{x \mid \exists\ i \in I, x \in A_i\}.$$

Negation of Quantified Statements

Consider the statement: "All math students are male." Its negation reads:
"It is not the case that all math students are male" or, equivalently, "There exists at least one math students who is a female (not male)"
Symbolically, using *M* to denote the set of math students, the above can be written as

$$\neg(\forall x \in M)\,(x \text{ is male}) \equiv (\exists\, x \in M)\,(x \text{ is not male})$$

or, when *p(x)* denotes "*x* is male,"

$$\neg(\forall x \in M)\, p(x) \equiv (\exists\, x \in M) \neg p(x)$$

Or $\neg \forall x\, p(x) \equiv \exists\, x \neg\, p(x).$

The above is true for any proposition *p(x)*. That is:

Theorem (DeMorgan):

(a) $\neg(\forall x \in A)p(x) \equiv (\exists\, x \in A)\neg p(x)$

That is

(1) It is not true that for all a ∈ *A* , *p* (a) is true.
(2) There exists an a ∈ *A* such that *p* (a) is false.

(b) $\neg(\exists\, x \in A)\, p(x) \equiv (\forall x \in A)\neg p(x).$

That is

(1) It is not true that for some a ∈ *A* , *p* (a) is true.
(2) For all a ∈ *A* , *p* (a) is false

Example :

(a) The following statements are negatives of each other:

"For all positive integers *n* we have *n* + 2 greater than 8"
"There exists a positive integer *n* such that *n* + 2 not greater than 8"

(b) The following statements are also negatives of each other:

"There exists a (living) person who is 150 years old"
"Every living person is not 150 years old".

Remark:

The expression $\neg p(x)$ has the obvious meaning:

> "The statement $\neg p(a)$ is true when $p(a)$ is false, and vice versa"

Previously, $\neg$ was used as an operation on statements; here $\neg$ is used as an operation on propositional functions.

Similarly, $p(x) \wedge q(x)$, read "$p(x)$ and $q(x)$," is defined by:

> "The statement $p(a) \wedge q(a)$ is true when $p(a)$ and $q(a)$ are true"

Similarly, $p(x) \vee q(x)$, read "$p(x)$ or $q(x)$," is defined by:

> "The statement $p(a) \vee q(a)$ is true when $p(a)$ or $q(a)$ is true"

Thus, in terms of truth sets:

(i) $\neg p(x)$ is the complement of $p(x)$.

(ii) $p(x) \wedge q(x)$ is the intersection of $p(x)$ and $q(x)$.

(iii) $p(x) \vee q(x)$ is the union of $p(x)$ and $q(x)$.