$$y\left(-\frac{\eta}{3z}\frac{\partial C_A}{\partial \eta}\right) = \frac{1}{z^{2/3}}\frac{\partial^2 C_A}{\partial \eta^2}$$

$$\frac{\partial^2 C_A}{\partial \eta^2} + \frac{1}{3}\eta^2\frac{\partial C_A}{\partial \eta} = 0 \Rightarrow \frac{d^2 C_A}{d\eta^2} + \frac{1}{3}\eta^2\frac{dC_A}{d\eta} = 0$$

$$P = \frac{dC_A}{d\eta} \quad , \quad \frac{dP}{d\eta} = \frac{d^2 C_A}{d\eta^2}$$

$$\frac{dP}{d\eta} + \frac{1}{3}\eta^2 P = 0 \Rightarrow \frac{dP}{P} + \frac{1}{3}\eta^2 d\eta = 0$$

$$\ln P + \frac{1}{9}\eta^3 - \ln A = 0 \Rightarrow P = A e^{-\eta^3/9}$$

$$\frac{dC_A}{d\eta} = A e^{-\eta^3/9} \Rightarrow \int_{C_A}^{0} dC_A = A\int_{\eta}^{\infty} e^{-\eta^3/9} d\eta$$

$$-C_A = A\int_{\eta}^{\infty} e^{-\eta^3/9} d\eta + B$$

B.C.1 $y = 0$ $C_A = C_{A0}$ $\eta = 0$

B.C.2 $y = \infty$ $C_A = 0$ $\eta = \infty$

Apply B.C.2 $\eta = \infty$ $C_A = 0$

$$0 = A\int_{\infty}^{\infty} e^{-\eta^3/9} d\eta + B \Rightarrow B = 0$$

$$\therefore C_A = -A\int_{\eta}^{\infty} e^{-\eta^3/9} d\eta$$

Apply B.C.1 $\eta = 0$ $C_A = C_{A0}$

$$C_{A0} = -A\int_{0}^{\infty} e^{-\eta^3/9} d\eta$$

This integration is the Gamma function (Γ).

Let $\beta = \frac{\eta^3}{9}$

$$d\beta = 3\left(\frac{\eta^2}{9}\right)d\eta \Rightarrow d\eta = \frac{1}{3}\left(\frac{\eta^2}{9}\right)^{-1}d\beta$$

$$C_{Ao} = -A\int_0^{\infty} e^{-\beta}\,\frac{1}{3}\left(\frac{\eta^2}{9}\right)^{-1}d\beta$$

Laplace Transforms :

Laplace transformation is a method to solve differential equations. Where ordinary and difference differential equations are converted to algebraic ones, while partial differential equations are converted to ordinary differential equations.

Definition :

Laplace transform is defined as the integral from 0 to ∞ of the function $f(x)$ and e^{-sx}. Where s is the Laplace operator and $\mathcal{L}$ is a designation for the transform operation.

$$\mathcal{L}\, f(x) = \int_0^\infty f(x) . e^{-sx} \, dx = f(s)$$

Laplace transform of some elementry functions :

$$1.\ \mathcal{L}\, K = \int_0^\infty e^{-sx} K \, dx = K \int_0^\infty e^{-sx} dx = K \left[\frac{e^{-sx}}{-s} \right]_0^\infty$$

$$= K \left[0 - \frac{1}{-s} \right] = \frac{K}{s}$$

$$\boxed{\mathcal{L}\, K = \frac{K}{s}}$$ where K is constant

$s > 0$

Example: $\mathcal{L}\, 1 = \frac{1}{s}$, $\mathcal{L}\, 10 = \frac{10}{s}$

$$2.\ \mathcal{L}\, x = \int_0^\infty e^{-sx} . x \, dx \quad , \quad \int u dv = u.v - \int v du$$

$$= x.\left(-\frac{e^{-sx}}{s}\right)\Bigg]_0^{\infty} - \int_0^{\infty} \frac{-e^{-sx}}{s}\, dx$$

$$\boxed{\begin{array}{ll} u = x & du = dx \\ dv = e^{-sx}\, dx & v = \dfrac{e^{-sx}}{-s} \end{array}}$$

$$= \cancel{-\frac{x}{s} e^{-\infty}}^{\,0} - \left(\cancel{-\frac{0}{s} e^{0}}^{\,0}\right) - \frac{1}{s^2}\, e^{-sx}\Bigg]_0^{\infty}$$

$$= \cancel{-\frac{1}{s^2} e^{-\infty}}^{\,0} - \left(-\frac{1}{s^2} e^{0}\right) = \frac{1}{s^2}$$

$$\boxed{\mathcal{L}\, x = \frac{1}{s^2}} \qquad s > 0$$

$$\mathcal{L}\, x^2 = \int_0^{\infty} x^2 . e^{-sx}\, dx \quad , \quad \int u\, dv = u.v - \int v\, du$$

$$= x^2 \left(\cancel{\frac{e^{-sx}}{-s}}^{\,0}\right)\Bigg]_0^{\infty} - \int_0^{\infty} 2x \left(\frac{e^{-sx}}{-s}\right) dx$$

$$\boxed{\begin{array}{ll} u = x^2 & du = 2x\, dx \\ dv = e^{-sx}\, dx & v = \dfrac{e^{-sx}}{-s} \end{array}}$$

$$= \frac{2}{s} \int_0^{\infty} x . e^{-sx}\, dx$$

$$= \frac{2}{s} . \frac{1}{s^2} = \frac{2}{s^3}$$

$$\boxed{\mathcal{L}\, x^2 = \frac{2}{s^3}} \qquad s > 0$$

$$\mathcal{L}\, x^n = \int_0^{\infty} x^n . e^{-sx}\, dx$$

let $z = sx$, $dz = s\, dx \Rightarrow dx = \frac{1}{s}\, dz$

$$= \int_0^{\infty} \frac{s^n}{s^n} x^n . \frac{s}{s} . e^{-sx}\, dx = \int_0^{\infty} \frac{z^n}{s^n} . e^{-z} \frac{dz}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^{\infty} z^n . e^{-z}\, dz = \frac{1}{s^{n+1}} \Gamma(n+1) = \frac{n!}{s^{n+1}}$$

$$\boxed{\mathcal{L}\, x^n = \frac{n!}{s^{n+1}}}$$

n is integer number

$s > 0$

Example: $\mathcal{L}x = \dfrac{1!}{s^{1+1}} = \dfrac{1}{s^2}$.

$$\mathcal{L}x^2 = \frac{2!}{s^{2+1}} = \frac{2}{s^3}.$$

$$\mathcal{L}x^3 = \frac{3!}{s^{3+1}} = \frac{6}{s^4}.$$

__Notes:__

A. $\Gamma n = (n-1)\,\Gamma(n-1)$, $\Gamma 1 = 1$ & $\Gamma\frac{1}{2} = \sqrt{\pi}$

Example: $\Gamma\frac{3}{2} = \overset{\frac{3}{2}-1}{\frac{1}{2}}\,\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$.

$$\Gamma\frac{5}{2} = \overset{\frac{5}{2}-1}{\frac{3}{2}}\,\Gamma\frac{3}{2} = \frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{3}{4}\sqrt{\pi}.$$

B. $\Gamma(n-1) = \dfrac{\Gamma n}{n-1}$

Example: $\Gamma\frac{-5}{2}$ $\quad n-1 = -\frac{5}{2} \Rightarrow n = -\frac{5}{2}+1 \Rightarrow n = -\frac{3}{2}$

$$\Gamma\frac{-5}{2} = \frac{\Gamma\frac{-3}{2}}{\frac{-5}{2}} = \frac{-2}{5}\,\Gamma\frac{-3}{2} = \frac{-2}{5}\,\frac{\Gamma\frac{-1}{2}}{-\frac{3}{2}}$$

$$= \left(\frac{-2}{5}\right)\left(\frac{-2}{3}\right)\Gamma\frac{-1}{2} = \left(\frac{-2}{5}\right)\left(\frac{-2}{3}\right)\frac{\Gamma\frac{1}{2}}{-\frac{1}{2}}$$

$$= \left(\frac{-2}{5}\right)\left(\frac{-2}{3}\right)\left(\frac{-2}{1}\right)\sqrt{\pi}.$$

3. $\mathcal{L}\, e^{ax} = \displaystyle\int_0^\infty e^{-sx}\cdot e^{ax}\cdot dx = \int_0^\infty e^{-(s-a)x}\cdot dx$

$$= \left[\frac{e^{-(s-a)x}}{-(s-a)}\right]_0^\infty = 0 - \frac{1}{-(s-a)} = \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\, e^{ax} = \frac{1}{s-a}}$$

Example: $\mathcal{L} e^{2x} = \dfrac{1}{s-2}$

$$\mathcal{L} e^{-3x} = \frac{1}{s+3}$$

4. $\mathcal{L} \sin ax = \int_0^\infty e^{-sx} . \sin ax . dx \Rightarrow \mathcal{L} \sin ax = \dfrac{a}{s^2+a^2}$

$$\mathcal{L} \cos ax = \int_0^\infty e^{-sx} . \cos ax . dx \Rightarrow \mathcal{L} \cos ax = \frac{s}{s^2+a^2}$$

$$e^{iax} = \cos ax + i \sin ax$$

$$\mathcal{L} e^{iax} = \mathcal{L} \cos ax + i \mathcal{L} \sin ax$$

$$\mathcal{L} e^{iax} = \frac{1}{s-ia} = \frac{1}{s-ia} . \frac{s+ia}{s+ia} \qquad i = \sqrt{-1},\ i^2 = -1$$

$$= \frac{s+ia}{s^2 - i^2a^2} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$$

Since, we know that the cosine is the real part and the sine is the imaginary part. Therefore,

$$\boxed{\mathcal{L} \cos ax = \frac{s}{s^2+a^2}}$$

$$\boxed{\mathcal{L} \sin ax = \frac{a}{s^2+a^2}}$$

Example: $\mathcal{L} \sin 2x = \dfrac{2}{s^2+4}$, $\mathcal{L} \cos 3x = \dfrac{s}{s^2+9}$

5. $\mathcal{L}(\sinh ax) = \mathcal{L}\left(\frac{e^{ax} - e^{-ax}}{2}\right) = \frac{1}{2}\left[\mathcal{L}\, e^{ax} - \mathcal{L}\, e^{-ax}\right]$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{s+a-(s-a)}{2(s-a)(s+a)}$$

$$= \frac{2a}{2(s^2-a^2)} = \frac{a}{s^2-a^2}$$

$$\boxed{\mathcal{L}(\sinh ax) = \frac{a}{s^2-a^2}}$$

$$\mathcal{L}(\cosh ax) = \mathcal{L}\left(\frac{e^{ax} + e^{-ax}}{2}\right) = \frac{1}{2}\left[\mathcal{L}\, e^{ax} + \mathcal{L}\, e^{-ax}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s+a+s-a}{2(s-a)(s+a)}$$

$$= \frac{2s}{2(s^2-a^2)} = \frac{s}{s^2-a^2}$$

$$\boxed{\mathcal{L}(\cosh ax) = \frac{s}{s^2-a^2}}$$

6. $\mathcal{L}\, x \cdot e^{-ax} = \int_0^\infty e^{-sx} . x . e^{-ax}\, dx = \int_0^\infty x . e^{-(s+a)x}\, dx$

$u = x \qquad du = dx$

$\int u\,dv = u.v - \int v\,du$,

$dv = e^{-(s+a)x}\,dx \qquad v = \frac{e^{-(s+a)x}}{-(s+a)}$

$$= x\,\frac{e^{-(s+a)x}}{-(s+a)} - \int_0^\infty \frac{e^{-(s+a)x}}{-(s+a)}\,dx$$

$$= 0 - \frac{1}{-(s+a)}\left[\frac{e^{-(s+a)x}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{s+a}\left[0 - \frac{1}{-(s+a)}\right]$$

$$= \frac{1}{s+a} \cdot \frac{1}{s+a} = \frac{1}{(s+a)^2}$$

$$\boxed{\mathcal{L}\, x \cdot e^{-ax} = \frac{1}{(s+a)^2}}$$

$f(x)$	$f(s)$
K	$\frac{k}{s}$
x	$\frac{1}{s^2}$
x^2	$\frac{2!}{s^3}$
x^n	$\frac{n!}{s^{n+1}}$
e^{ax}	$\frac{1}{s-a}$
$\sin ax$	$\frac{a}{s^2+a^2}$
$\cos ax$	$\frac{s}{s^2+a^2}$
$\sinh ax$	$\frac{a}{s^2-a^2}$
$\cosh ax$	$\frac{s}{s^2-a^2}$
$x e^{ax}$	$\frac{1}{(s-a)^2}$

Rules of Laplace Transforms :

The Laplace transform is a linear transform by which is meant that:

1. The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms.

$$\mathcal{L}\{f(x) \pm g(x)\} = \mathcal{L}\, f(x) \pm \mathcal{L}\, g(x)$$

2. The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression.

$$\mathcal{L}\{K f(x)\} = K\, \mathcal{L}\, f(x)$$

Example: Find Laplace transform of the followings:

1. $\mathcal{L}\{2e^{-x} + x\} = 2\mathcal{L}\, e^{-x} + \mathcal{L}\, x = \frac{2}{s+1} + \frac{1}{s^2}$

$$= \frac{2s^2 + (s+1)}{s^2(s+1)} = \frac{2s^2 + s + 1}{s^2(s+1)}$$

2. $\mathcal{L}\{\sin 3x + \cos 3x\} = \mathcal{L}\sin 3x + \mathcal{L}\cos 3x$

$$= \frac{3}{s^2+3^2} + \frac{s}{s^2+3^2} = \frac{3+s}{s^2+9}$$

3. $\mathcal{L}\{3x^3 + \sin x\} = 3\mathcal{L}\, x^3 + \mathcal{L}\sin x$

$$= 3\, \frac{3!}{s^{3+1}} + \frac{1}{s^2+1^2} = \frac{18}{s^4} + \frac{1}{s^2+1}$$

$$\frac{18(s^2+1)+s^4}{s^4(s^2+1)} = \frac{s^4+18s^2+18}{s^4(s^2+1)}$$

4. $\mathcal{L}\{4e^{2x} + 3\cosh 4x\} = 4\mathcal{L}\,e^{2x} + 3\mathcal{L}\cosh 4x$

$$= 4\,\frac{1}{s-2} + 3\,\frac{s}{s^2-4^2} = \frac{4}{s-2} + \frac{3s}{s^2-16}$$

The First Shifting Theorem :

If $\mathcal{L}\,f(x) = f(s)$ then $\mathcal{L}\{e^{-ax} f(x)\} = f(s+a)$

$$\mathcal{L}\{e^{-ax} f(x)\} = \int_0^\infty e^{-ax} f(x)\, e^{-sx}\, dx = \int_0^\infty f(x)\, e^{-(s+a)x}\, dx = f(s+a)$$

Example : Find Laplace transform of the followings :

1. $\mathcal{L}\{e^{3x} \cos 4x\}$

$$\mathcal{L}\cos 4x = \frac{s}{s^2+16}$$

$$\mathcal{L}\{e^{3x} \cdot \cos 4x\} = \frac{s-3}{(s-3)^2+16} = \frac{s-3}{s^2-6s+25}$$

2. $\mathcal{L}\{e^{-2x} \cdot \sin 3x\}$

$$\mathcal{L}\sin 3x = \frac{3}{s^2+9}$$

$$\mathcal{L}\{e^{-2x} \cdot \sin 3x\} = \frac{3}{(s+2)^2+9} = \frac{3}{s^2+4s+13}$$

3. $\mathcal{L}\{e^{3x}(x^2+4)\}$

$$\mathcal{L}\{x^2+4\} = \mathcal{L}x^2 + \mathcal{L}4 = \frac{2}{s^3} + \frac{4}{s}$$

$$\mathcal{L}\{e^{3x}(x^2+4)\} = \frac{2}{(s-3)^3} + \frac{4}{(s-3)}$$

$$= \frac{2+4(s-3)^2}{(s-3)^3} = \frac{2+4(s^2-6s+9)}{(s-3)^3} = \frac{4s^2-24s+38}{(s-3)^3}$$

$f(x)$	$f(s)$
$e^{ax}.x^n$	$\frac{n!}{(s-a)^{n+1}}$
$e^{ax}.\sin bx$	$\frac{b}{(s-a)^2+b^2}$
$e^{ax}.\cos bx$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{ax}.\sinh bx$	$\frac{b}{(s-a)^2-b^2}$
$e^{ax}.\cosh bx$	$\frac{s-a}{(s-a)^2-b^2}$

4. $\mathcal{L}\{x.\cosh 3x\} = \mathcal{L}\left\{x.\frac{e^{3x}+e^{-3x}}{2}\right\}$

$$= \frac{1}{2}\{\mathcal{L}x e^{3x} + \mathcal{L}x e^{-3x}\}$$

$$= \frac{1}{2}\left\{\frac{1}{(s-3)^2} + \frac{1}{(s+3)^2}\right\}$$