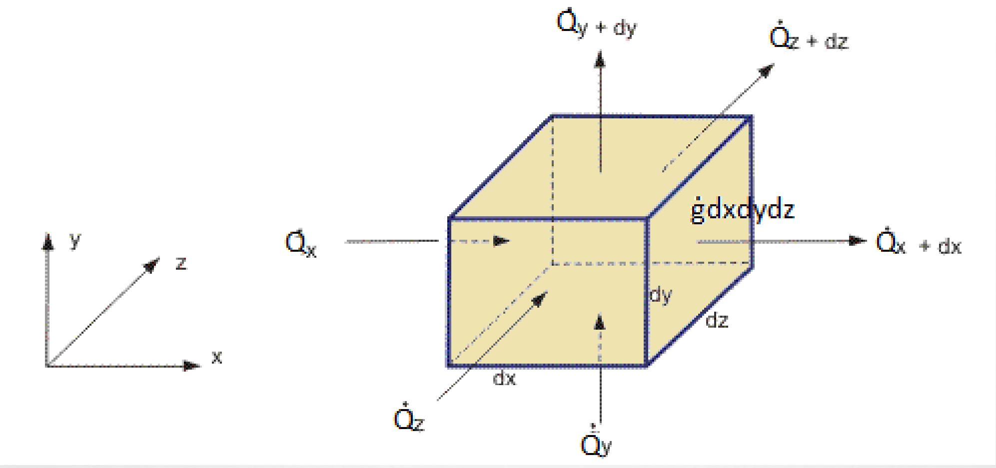
ONE-DIMENSIONAL HEAT CONDUCTION EQUATION

PROOF. DR. MAJID H. MAJEED

THREE DIMENSIONAL EQUATION



$$\begin{pmatrix} Rate\,of\ Heat \\ Conduction\,at \\ x, y, and\ z \end{pmatrix} - \begin{pmatrix} Rate\,of\ Heat \\ Conduction\,at \\ x+dx, y+dy, \\ and\ z+dz \end{pmatrix} + \begin{pmatrix} Rate\,of\ Heat \\ Generation \\ inside\,the \\ Element \end{pmatrix} = \begin{pmatrix} Rate\,of\ Change \\ of\ the\,Energy \\ Content\,in\,the \\ Element \end{pmatrix}$$

$$\left(\dot{Q}_x - \dot{Q}_{x+dx}\right) + \left(\dot{Q}_y - \dot{Q}_{y+dy}\right) + \left(\dot{Q}_z - \dot{Q}_{z+dz}\right) + \dot{G}_{elem} = \frac{d\dot{E}_{elem}}{d\tau}$$

THREE DIMENSION CONDUCTION

$$\dot{Q}_x = -kA_x \frac{dT}{dx}, \qquad \dot{Q}_{x+dx} = -kA_x \frac{dT}{dx} - \frac{d}{dx}\left(kA_x \frac{dT}{dx}\right)dx$$

$$\dot{Q}_y = -kA_y \frac{dT}{dy}, \qquad \dot{Q}_{y+dy} = -kA_y \frac{dT}{dy} - \frac{d}{dy}\left(kA_y \frac{dT}{dy}\right)dy$$

$$\dot{Q}_z = -kA_z \frac{dT}{dz}, \qquad \dot{Q}_{z+dz} = -kA_z \frac{dT}{dz} - \frac{d}{dz}\left(kA_z \frac{dT}{dz}\right)dz$$

And the heat generation is $\dot{G}_{elem} = \dot{g}dV = \dot{g}dxdydz$

Where A_x=dydz, A_y=dzdx, A_z=dxdy

dV=dxdydz=$A_x$dx=$A_y$dy=$A_z$dz

And the change of energy content in the element is $\dfrac{dE_{elem}}{d\tau} = mC\dfrac{dT}{d\tau} = \rho C dV \dfrac{dT}{d\tau}$

THREE DIMENSIONAL CONDUCTION

$$\dot{Q}_x - \dot{Q}_{x-dx} = -kA_x \frac{dT}{dx} - \left\{ -kA_x \frac{dT}{dx} - \frac{d}{dx}\left(k\frac{dT}{dx}\right)A_x dx \right\} = \frac{d}{dx}\left(k\frac{dT}{dx}\right)A_x dx$$

$$\dot{Q}_x - \dot{Q}_{x-dx} = \frac{d}{dx}\left(k\frac{dT}{dx}\right)dxdydz$$

And $$\dot{Q}_y - \dot{Q}_{y-dy} = \frac{d}{dy}\left(k\frac{dT}{dy}\right)dxdydz$$

And $$\dot{Q}_z - \dot{Q}_{z-dz} = \frac{d}{dz}\left(k\frac{dT}{dz}\right)dxdydz$$

By substituting these terms in eq.(2.2), we find that

$$\left\{ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} \right\} dxdydz = \rho C \frac{\partial T}{\partial \tau} dxdydz$$

$$\left\{ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} \right\} = \rho C \frac{\partial T}{\partial \tau}$$

$$\alpha = k / \rho C$$

THREE DIMENSIONAL CONDUCTION

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

And it can reduced under specific conditions to the following forms

1- Steady state conduction (The Poisson Equation)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = 0$$

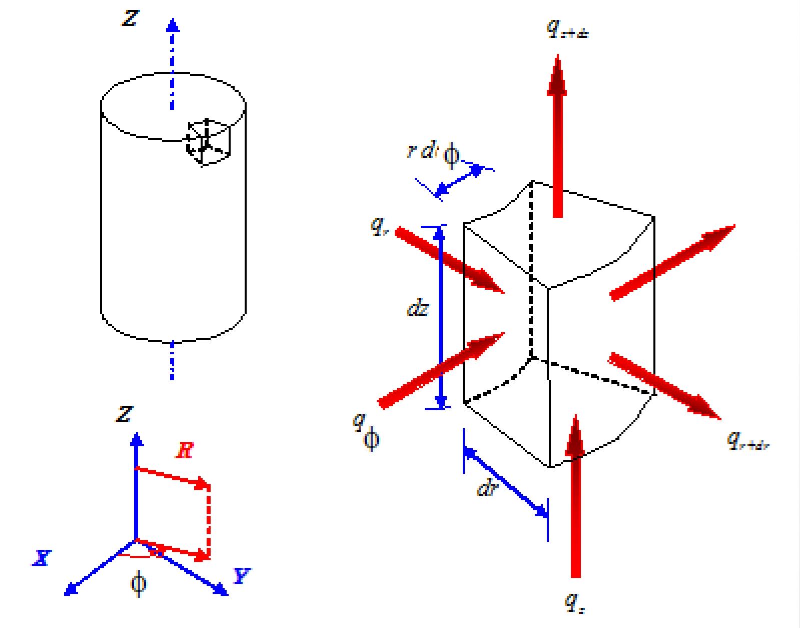
2- Transient, and with no heat generation (Diffusion Equation)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

3- Steady state, and with no heat generation (The Laplace Equation)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

THREE DIMENSIONAL CONDUCTION CYLINDRICAL COORDINATE



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

CYLINDRICAL COORDINATE

This equation can be reduced under specific conditions to the following forms

1- Steady state (Poisson Equation)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)+\frac{\dot{g}}{k}=0$$

2- Transient, And No Heat Generation(The Diffusion Equation)

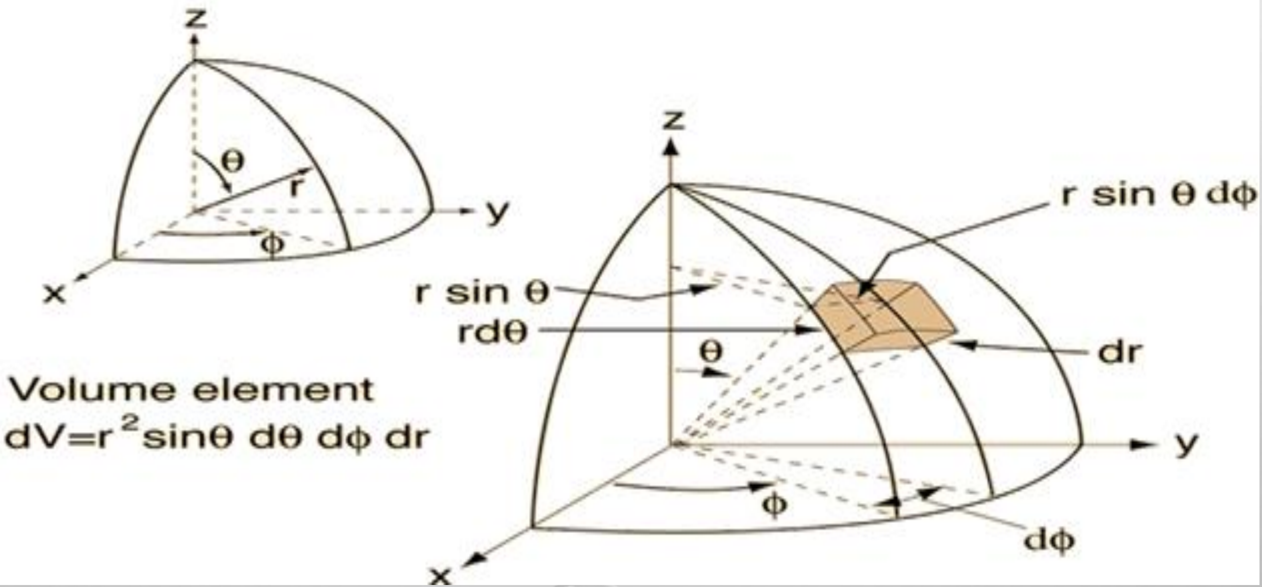
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

3- Steady State, And No Heat Generation(The Laplace Equation)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)=0$$

SPHERICAL COORDINATE

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\frac{\partial T}{\partial\theta}\right)+\dot{g}=\rho C\frac{\partial T}{\partial\tau}$$



ONE-DIMENSINAL HEAT CONDUCTION EQUATION
HEAT CONDUCTION EQUATION IN A LARGE PLANE WALL

$$\left\{\begin{matrix}\text{Rate of Heat}\\ \text{Conduction}\\ \text{at } x\end{matrix}\right\} - \left\{\begin{matrix}\text{Rate of Heat}\\ \text{Conduction}\\ \text{at } x+\Delta x\end{matrix}\right\} + \left\{\begin{matrix}\text{Rate of Heat}\\ \text{Generation}\\ \text{Inside the Element}\end{matrix}\right\} = \left\{\begin{matrix}\text{Rate of Change of}\\ \text{The Energy Stored}\\ \text{in The Element}\end{matrix}\right\}$$

Or
$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{G}_{elem} = \frac{\Delta E_{elem}}{\Delta \tau}$$

The energy change in stored or contained in the element can be found as

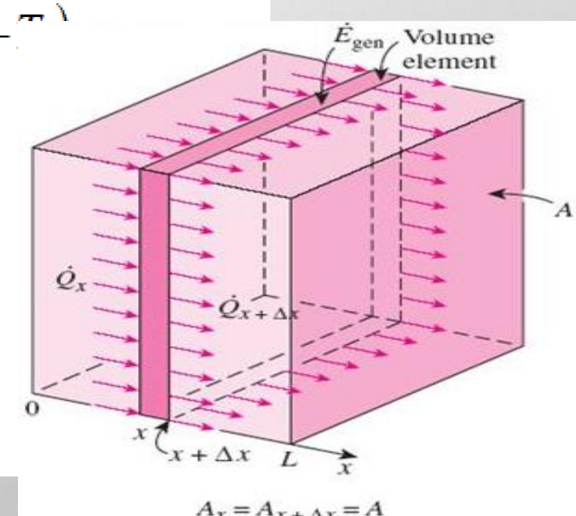
$$\Delta E_{elem} = E_{\tau+\Delta\tau} - E_{\tau} = mC\left(T_{\tau+\Delta\tau} - T_{\tau}\right) = \rho C A \Delta x\left(T_{\tau+\Delta\tau} - \right.$$

By substituting into this equation into eq.(2.20)

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{g} A \Delta x = \rho C A \Delta x \frac{T_{\tau+\Delta\tau} - T_{\tau}}{\Delta \tau}$$

And by dividing eq.(2.12) by (AΔx) we obtain

$$-\frac{1}{A}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} + \dot{g} = \rho C \frac{T_{\tau+\Delta\tau} - T_{\tau}}{\Delta \tau}$$



ONE-DIMENSIONAL CONDUCTION EQUATION

$$-\frac{1}{A}\lim_{\Delta x\to 0}\frac{\dot{Q}_{x+\Delta x}-\dot{Q}_x}{\Delta x}+\dot{g}=\rho C\lim_{\Delta\tau\to 0}\frac{T_{\tau+\Delta\tau}-T_\tau}{\Delta\tau}$$

Then
$$-\frac{1}{A}\frac{\partial\dot{Q}}{\partial x}+\dot{g}=\rho C\frac{\partial T}{\partial\tau}$$

And from Fourier equation we know that $\left(\dot{Q}=-Ak\frac{dT}{dx}\right)$, and by substituting that in above equation we get

$$-\frac{1}{A}\frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right)+\dot{g}=\rho C\frac{\partial T}{\partial\tau}$$

It notes that the area A is constant for a plane wall. The equation of the transient heat conduction in one-dimension a plane wall becomes

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{g}=\rho C\frac{\partial T}{\partial\tau}$$

CONDUCTION EQUATION IN PLANE WALL

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad (2.25)$$

Where the thermal diffusivity $(\alpha = k/\rho C)$ is a property of the material represents how fas heat diffuse through material. Eq.(2.25) under specified conditions can be reduced to forms.

1- Steady state: $\partial/\partial\tau = 0$

$$\frac{d^2T}{dx^2} + \frac{\dot{g}}{k} = 0$$

2- Transient, with no heat generation $\dot{g} = 0$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

3- Steady state, with no heat generation ($\partial/\partial\tau = 0, \dot{g} = 0$)

$$\frac{d^2T}{dx^2} = 0$$

HEAT CONDUCTION EQUATION IN A LONG CYLINDER

$$\left\{\begin{matrix} Rate\ of\ Heat \\ Conduction \\ at\ r \end{matrix}\right\} - \left\{\begin{matrix} Rate\ of\ Heat \\ Conduction \\ at\ r+\Delta r \end{matrix}\right\} + \left\{\begin{matrix} Rate\ of\ Heat \\ Generation \\ Inside\ the\ Element \end{matrix}\right\} = \left\{\begin{matrix} Rate\ of\ Change\ of \\ The\ Energy\ Stored \\ in\ The\ Element \end{matrix}\right\}$$

Or $$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{G}_{elem} = \frac{\Delta E_{elem}}{\Delta \tau}$$

The change in the heat stored of the element can be expressed as

$$\Delta E_{elem} = E_{\tau+\Delta\tau} - E_{\tau} = \rho C A \Delta r \left(T_{\tau+\Delta\tau} - T_{\tau}\right)$$

By substituting into eq.(2.30) we obtain

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{g} A \Delta r = \rho C A \Delta r \frac{T_{\tau+\Delta\tau} - T_{\tau}}{\Delta \tau}$$

By dividing by AΔr gives

$$-\frac{1}{A}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} + \dot{g} = \rho C \frac{T_{\tau+\Delta\tau} - T_{\tau}}{\Delta \tau}$$

And taking the limit as $\Delta r \to 0$ and $\Delta \tau \to 0$

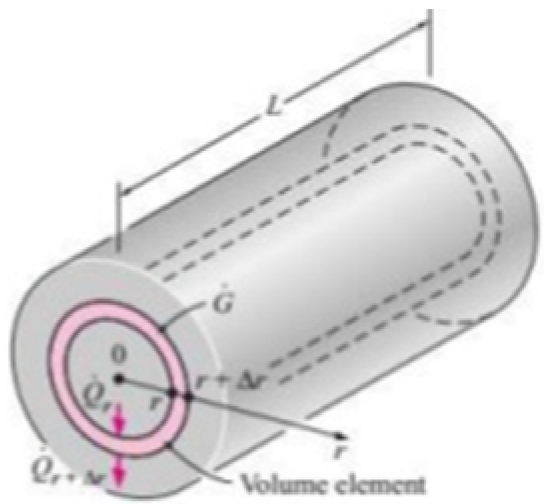
$$-\frac{1}{A}\lim_{\Delta r\to 0}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} + \dot{g} = \rho C \lim_{\Delta\tau\to 0}\frac{T_{\tau+\Delta\tau} - T_{\tau}}{\Delta \tau}$$

HEAT CONDUCTION EQUATION IN A LONG CYLINDER

Fourier equation give us that $\left(\dot{Q} = -Ak\dfrac{dT}{dr}\right)$, and $A = 2\pi rL$. Substituting that in above equation we get that

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(-kr\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C\frac{\partial T}{\partial \tau} \qquad (2.34)$$

The one dimensional equation of transient heat conduction in a cylinder becomes:-



HEAT CONDUCTION EQUATION IN A LONG CYLINDER

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\dot{g}=\rho C\frac{\partial T}{\partial \tau} \qquad (2.35)$$

For constant thermal conductivity, eq.(2.35) can be reduced to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad (2.36)$$

α is the property which known as thermal diffusivity as defined before. Eq.(2.36) under the specified conditions reduces to the following forms.

1- Steady state: $\partial/\partial\tau = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{g}}{k}=0 \qquad (2.37)$$

2- Transient, with No heat generation $\dot{g}=0$

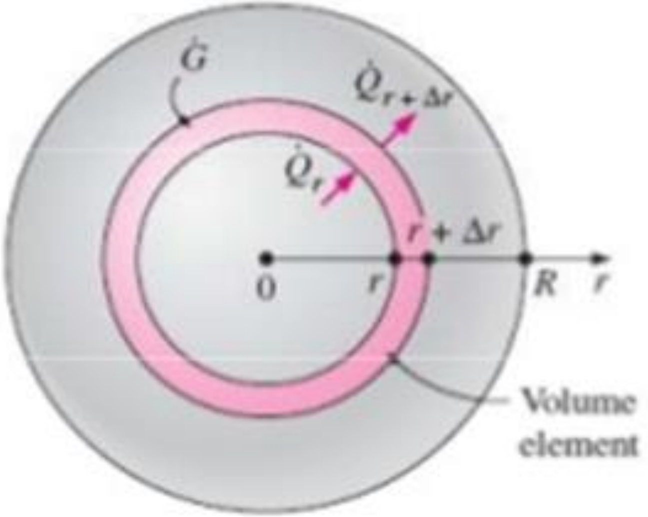
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad (2.38)$$

3- Steady state, with No generation heat $(\partial/\partial\tau = 0, \dot{g}=0)$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=0 \qquad (2.39)$$

HEAT CONDUCTION EQUATION IN A SPHERE

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\dot{g}=\rho C\frac{\partial T}{\partial \tau}$$



HEAT CONDUCTION EQUATION IN A SPHERE

For the constant thermal conductivity, eq.(2.40) reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

This equation can be reduced for the following conditions

1- steady State heat conduction $\partial/\partial\tau=0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{g}}{k}=0$$

2- Transient, no heat generation $\dot{g}=0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

3- Steady state, No generation heat $(\partial/\partial\tau=0\,,\dot{g}=0)$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=0$$

GENERAL HEAT CONDUCTION EQUATION OF ONE-DIMENSION

2.2.4 General Heat Conduction Equation Of One-Dimension

The one-dimensional equations of transient heat conduction for a plane wall, Cylinder, and sphere can be represented in a uniform equation as

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n k\frac{\partial T}{\partial r}\right)+\dot{g}=\rho C\frac{\partial T}{\partial \tau} \qquad (2.45)$$

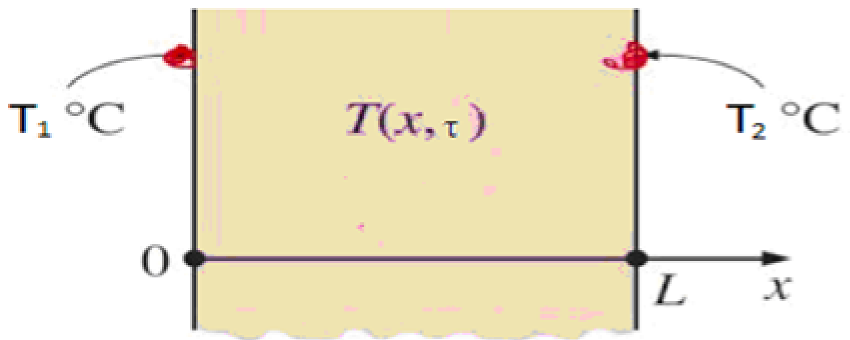
We can take n=0 for a plane wall. And take n=1 for a cylindrical wall. Also take n=2 for a spherical wall. For the case of a plane wall, r is always replaced by x. This equation is also simplified for a cases of no heat generation and steady state as discussed before.

INITIAL AND BOUNDARY CONDITIONS

- THERE ARE MANY TYPE OF BOUNDARY CONDITIONS

- **SPECIFIED TEMPERATURE BOUNDARY CONDITIONS**

$$T(0,\tau) = T_1 = 0^{o}C \quad \text{and} \quad T(L,\tau) = T_2 = 100^{o}C$$



$$T(0, \tau) = T_1$$
$$T(L, \tau) = T_2$$

INITIAL AND BOUNDARY CONDITIONS

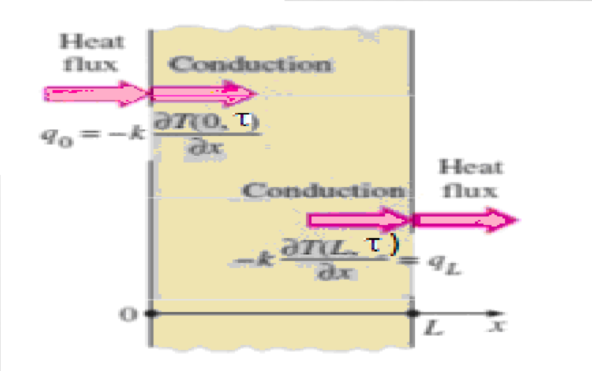
- **SPECIFIED HEAT FLUX BOUNDARY CONDITION**

$$\begin{Bmatrix} External\ Heat \\ Supply, \dot{q}_o W/m^2 \end{Bmatrix} = \begin{Bmatrix} Heat\ Flow\ by\ Conduction \\ Into\ the\ Body\ at\ x = 0 \end{Bmatrix}$$

That is, for example $\dot{q}_o = 100 W/m^2 = -k\left.\frac{\partial T(x,\tau)}{\partial x}\right|_{\substack{evaluated \\ at\ x=0}}$

Or, more compactly

$$-k\left.\frac{\partial T(x,\tau)}{\partial x}\right|_{x=0} = \dot{q}_o$$



INITIAL AND BOUNDARY CONDITIONS

Next, We consider a heat supply into the body at the rate of $\left|\dot{q}_L\right. = 200\,W/m^2$ through the boundary surface at x=L, as illustrated in Fig. 2.9. The energy balance at the surface at x=L.

$$\begin{Bmatrix} Heat\ Flow\ by\ Conduction \\ Into\ the\ Body\ at\ x=L \end{Bmatrix} = \begin{Bmatrix} External\ Heat\ Supply, \dot{q}_L \\ W/m^2\ into\ the\ Body\ at\ x=L \end{Bmatrix}$$

That is

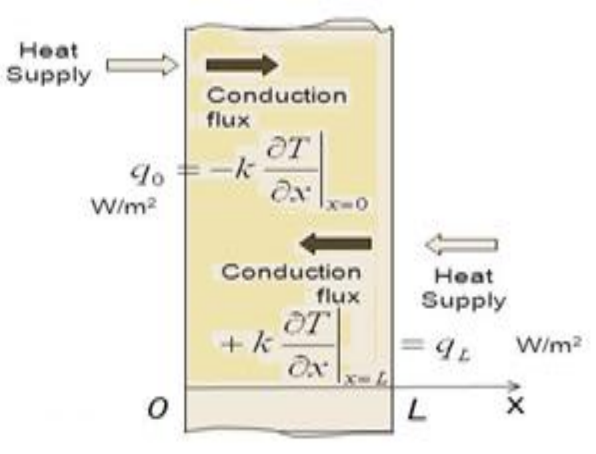
$$-k\frac{\partial(x,\tau)}{\partial x}\bigg|_{x=L} = -\dot{q}_L = -200\,W/m^2$$

Note that, if the direction of $\dot{q}_L$ is in the negative direction of x, so

$$k\frac{\partial(x,\tau)}{\partial x}\bigg|_{x=L} = \dot{q}_L \qquad (2.47)$$

$$-k\frac{\partial(x,\tau)}{\partial(x)} = \dot{q}_o \quad \text{at } x=0$$

$$k\frac{\partial(x,\tau)}{\partial(x)} = \dot{q}_L \quad \text{at } x=L$$



INITIAL AND BOUNDARY CONDITIONS

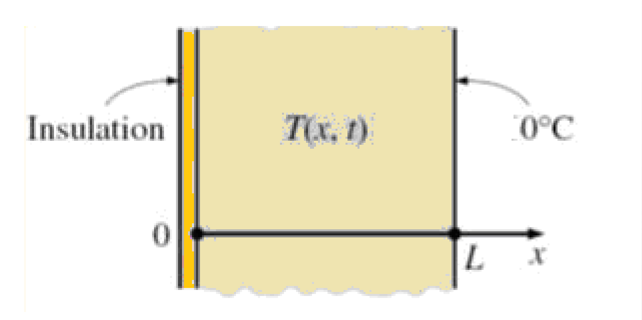
Insulated Boundary

$$q_o = 0$$

$$-k\frac{dT}{dx} = 0$$

$$k \neq 0$$

$$\frac{dT}{dx} = 0$$



$$\dot{q}_o = -k\frac{\partial T(x,\tau)}{\partial x}\bigg|_{x=0} = 0$$

$$\dot{q}_o = -k\frac{\partial T(x,\tau)}{\partial x}\bigg|_{x=0} = 0$$

INITIAL AND BOUNDARY CONDITIONS

Thermal Symmetry

$$\frac{dT}{dx} = 0$$